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SIMPLE TWO AND THREE DIMENSIONAL CASES IN THE THEORY OF CONSOLIDATION OF SOILS

By Nabor Carrillo[1]

When a fully saturated soil is suddenly loaded, the surcharge is initially balanced entirely by excess pressures created in the water. As time goes on, there is a gradual transfer of load from the water onto the solid particles. This process is known as "consolidation" of the soil and was first analyzed by Karl Terzaghi (1).

The equation governing the process of consolidation is the following:

$$\frac{\partial u}{\partial t} = c_v \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y_2} + \frac{\partial^2 u}{\partial z^2} \right) \quad \text{.....(I)}$$

wherein u is the excess hydrostatic pressure at any point (x, y, z) at any time t, and c_v is the so-called coefficient of consolidation. This coefficient of consolidation depends on the physical characteristics of the soil and on the state of deformation resulting from the applied load.

Equation (I) is identical with the differential equation of heat flow (2). If u is interpreted as temperature and c_v as the diffusivity constant, the solutions of the equation of consolidation can be utilized in the Theory of Conduction of Heat, and viceversa. Several solutions of heat conduction problems have in fact provided immediate answer to problems in the field of Soil Mechanics.

The purpose of this paper is to show how some familiar one and two dimensional solutions of the differential equation of consolidation (or of the analogous heat flow) can easily be combined to furnish simple solution for several important two and three dimensional problems.

Two cases will be considered here. The first one is presented primarily for the sake of simplicity in the discussion, although practical applications of it have been suggested. The solution to this first case is well known (2) and serves as an illustration. The second case is related to a practically important problem, the investigation of which led to the analysis herein presented.

Case I. Consolidation of an infinite rectangular prism of saturated soil which is loaded suddenly and is permitted to drain at all its faces. The initial excess hydrostatic pressure is constant at all points within the prism, and zero at the boundary faces.

Case II. Consolidation of a right circular cylinder of saturated soil which is loaded suddenly and is permitted to drain at all its surface. The initial excess hydrostatic pressure is constant at all points within the cylinder (which may be solid or hollow), and zero at the boundary surfaces.

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In finding the solution for the two problems as stated, the following theorems will be useful:

THEOREM 1. If $u_1 = f_1(x, t)$ is a solution of the linear flow equation

$$\frac{\partial u}{\partial t} = c_v \frac{\partial^2 u}{\partial x^2} \quad \text{.....(a)}$$

and $u_2 = f_2(y, t)$ is a solution of the linear flow equation

$$\frac{\partial u}{\partial t} = c_v \frac{\partial^2 u}{\partial y^2} \quad \text{.....(b)}$$

then $u = u_1 u_2$ is necessarily a solution of the two dimensional flow equation

$$\frac{\partial u}{\partial t} = c_v \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad \text{.....(c)}$$

To prove the Theorem, let us substitute the function $u = u_1 u_2$ in equation (c):

$$\frac{\partial (u_1 u_2)}{\partial t} = c_v \left\{ \frac{\partial^2 (u_1 u_2)}{\partial x^2} + \frac{\partial^2 (u_1 u_2)}{\partial y^2} \right\}$$

or

$$u_1 \frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_1}{\partial t} = c_v \left(u_2 \frac{\partial^2 u_1}{\partial x^2} + u_1 \frac{\partial^2 u_2}{\partial y^2} \right) \quad \text{.....(d)}$$

since u_1 does not depend on y, and u_2 does not depend on x.

Inserting in the left side of equation (d) the values given in (a) and (b), the following identity is obtained

$$u_1 c_v \frac{\partial^2 u_2}{\partial y^2} + u_2 c_v \frac{\partial^2 u_1}{\partial x^2} = c_v \left(u_2 \frac{\partial^2 u_1}{\partial x^2} + u_1 \frac{\partial^2 u_2}{\partial y^2} \right)$$

which proves the Theorem.

THEOREM 2. If $u_1 = f_1(r, t)$ is a solution of the symmetrical two dimensional flow equation

$$\frac{\partial u}{\partial t} = c_v \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right)$$

and $u_2 = f_2(z, t)$ is a solution of the linear flow equation

$$\frac{\partial u}{\partial t} = c_v \frac{\partial^2 u}{\partial z^2}$$

then $u = u_1 u_2$ is necessarily a solution of the three dimensional case of flow, symmetrical with respect to the z axis.

By substituting the function $u = u_1 u_2$ in the equation

$$\frac{\partial u}{\partial t} = c_v \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} \right)$$

the Theorem is readily proved.

Consider now the problem of the infinite prism, consolidating under the conditions specified above.

Assume that the faces of the prism are parallel to the xz and yz planes of a rectangular coordinate system. Imagine, initially, that the faces parallel to the xz plane are impervious, so that the flow is parallel to the x axis. The solution is well known and may briefly be written thus:

$$\frac{u_1}{u_0} = \varphi_1(x, t)$$

wherein u_1 is the excess hydrostatic pressure at a point of abscissa x, at a time t, u_0 is the initial excess hydrostatic pressure, and φ_1 is a function satisfying the following conditions:

a) At $t = 0$, $\varphi_1 = 1$ for all values of x within the prism.

b) At the drainage surfaces, $\varphi_1 = 0$ for all values of t.

Imagine now that the faces parallel to the yz plane are impervious, so that the flow is parallel to the y axis. The solution in this case is

$$\frac{u_2}{u_0} = \varphi_2(y, t)$$

wherein φ_2 satisfies the conditions:

a) At $t = 0$, $\varphi_2 = 1$ for all values of y within the prism.

b) At the drainage surfaces, $\varphi_2 = 0$ for all values of t.

It will be proved now that

$$\frac{u}{u_0} = \varphi_1(x, t) \cdot \varphi_2(y, t) \qquad (1)$$

is the solution of the problem when the soil drains at all four faces.

By Theorem 1, equation (1) satisfies the differential equation

$$\frac{\partial u}{\partial t} = c_v \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

Furthermore:

a) At $t = 0$, $\frac{u}{u_0} = 1$ for all points within the prism, and

b) At the boundary surfaces, $\frac{u}{u_0} = 0$ for all values of t.

Therefore, equation (1) satisfies the initial and boundary conditions, and $\frac{u}{u_0} = \frac{u_1}{u_0} \cdot \frac{u_2}{u_0}$ is the solution of the problem.

The preceding result may be stated as follows:

At any point within the prism which is consolidating under the specified conditions, the ratio between the excess hydrostatic pressure at any time, and the initial hydrostatic pressure, is the product of the two ratios obtained by considering first flow perpendicular to two faces only, and then flow perpendicular to the other faces only.

Furthermore, it is clear that the property just stated can be rewritten in terms of the average excess hydrostatic pressure in the prism, instead of the pressure at a particular point.

Through similar reasoning based on Theorem 2, the following property is established for the cylinder:

At any point within the cylinder which is consolidating under specified conditions, the ratio between the excess hydrostatic pressure at any time, and the

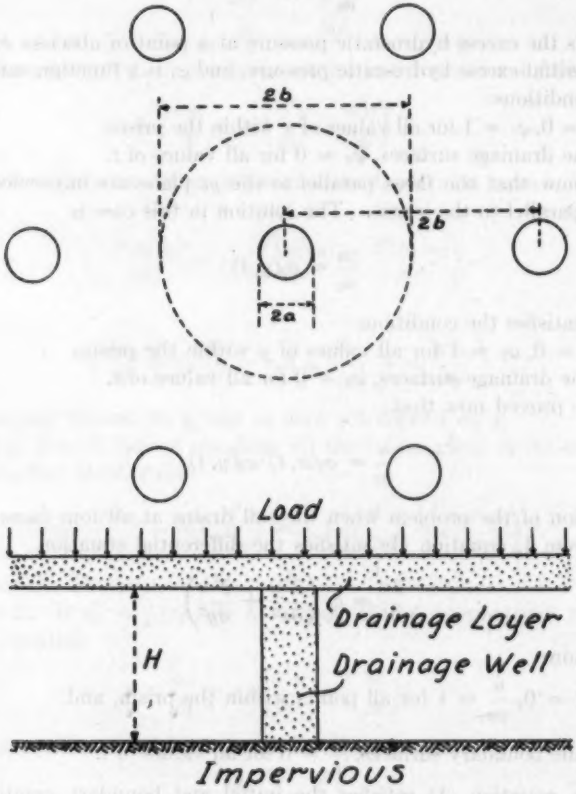

FIG. 1

initial excess pressure, is the product of the two ratios obtained by considering first drainage at the bases of the cylinder only, and then drainage at the cylindrical surfaces only. And the corresponding property connected with the average excess hydrostatic pressure within the prism is also true.

To illustrate the conclusions just reached, an important practical problem will be mentioned.

A system of wells is provided to accelerate the consolidation of a compressible

soil stratum, as shown in the diagram. It is required to find the rate of consolidation of the stratum.

If the top surface of the soil is initially assumed to be impervious, the problem is approximately reduced to that of an infinite right circular cylinder under the following conditions:

a) Impervious surface at $r = b$;

b) Drainage surface at $r = a$.

The solution of this problem was obtained by L. Rendulic (3). By combining Rendulic's solution with the well known solution for a stratum of depth H draining at the top surface only, the proposed problem is readily solved.

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DIFFERENTIAL EQUATION OF A SLIDING SURFACE IN AN IDEAL SATURATED PLASTIC SOIL

By Nabor Carrillo[1]

The relationship between the stresses acting on a failure surface in a plastic soil and the geometric characteristics of the surface may be expressed by a differential equation, provided the law of failure of the material is known.

The mathematical treatment of the problem is greatly simplified if the sliding surface is cylindrical. The analysis is then two dimensional and the sliding surface may be considered as a line in a plane.

Kötter (1) obtained the solution for the two dimensional case assuming that the law of failure of the material is given by the equation

$$t = n \tan \phi$$

wherein (n, t) are the normal and shearing components of stress in the plane of failure and ϕ is the angle of internal friction.

Jaky (2) proved that Kötter's differential equation is also valid for cohesive materials.

A fundamental limitation of Kötter's equation, when applied to a soil, is that the presence of water is ignored altogether. And it is a recognized fact in Soil Mechanics that the state of stress in the water is most important in connection with the stability of saturated soils.

Kötter's equation will be modified to include the effect of water pressures.

The stresses in a small element of a soil are shown in figure (1). (m', n', t') are the total stresses (including the effect of water pressure); ds is an element of a sliding line, and 0 is the center of curvature corresponding to the arc ds.

It is assumed that the only body force is the weight of soil plus water. That is, the velocity of percolation of water through the soil is considered small enough so that frictional body forces may be neglected.

The normal stresses on the sides of the element are assumed to consist of an effective and a neutral part. If the water is in a state of equilibrium, the neutral stresses increase in simple proportion to depth. On the other hand, if the water percolates through the voids of the soil, the neutral stresses change according to a law which is determined by the flownet. The following investigation is based on the latter assumption, which represents the general case.

Equilibrium of the element and continuity of stresses require that

$$\left.\begin{aligned} &\frac{\partial n'}{\partial R} - \frac{1}{R} \cdot \frac{\partial t'}{\partial \alpha} + \frac{n' - m'}{R} - \gamma \cos \alpha = 0 \\ &\frac{1}{R} \cdot \frac{\partial m'}{\partial \alpha} - \frac{\partial t'}{\partial R} - \frac{2t'}{R} + \gamma \cos \alpha = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (1)$$

[1] Fellow of the John Simon Guggenheim Memorial Foundation; Graduate School of Engineering, Harvard University.

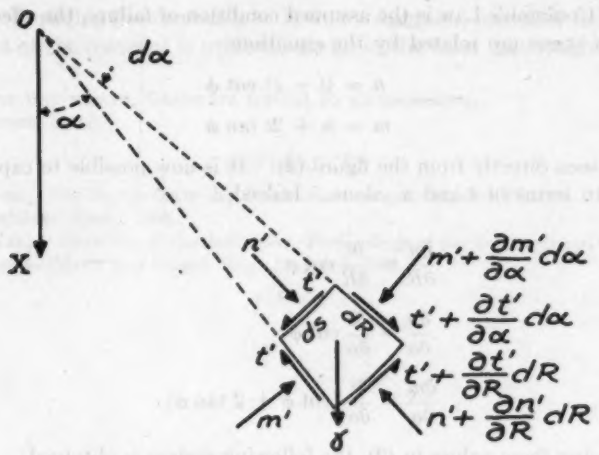

FIG. 1. Stresses on a Differential Element

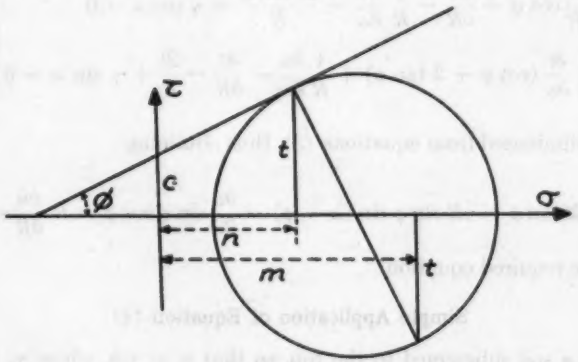

FIG. 2. Coulomb's Law of Failure

Now, let u be the neutral stress (water pressure) at the element under consideration. The effective components of stress (m, n, t) are given by the equations

$$m = m' - u$$
$$n = n' - u$$
$$t = t'$$

so that, in terms of effective stresses, equations (1) are

$$\left.\begin{aligned} &\frac{\partial n}{\partial R} + \frac{\partial u}{\partial R} - \frac{1}{R}\frac{\partial t}{\partial \alpha} + \frac{n-m}{R} - \gamma \cos \alpha = 0 \\ &\frac{1}{R}\cdot\frac{\partial m}{\partial \alpha} + \frac{1}{R}\frac{\partial u}{\partial \alpha} - \frac{\partial t}{\partial R} - \frac{2t}{R} + \gamma \sin \alpha = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (2)$$

Now, if Coulomb's Law is the assumed condition of failure, the effective components of stress are related by the equations

$$n = (t - c)\cot\phi$$

$$m = n + 2t\tan\phi$$

as can be seen directly from the figure (2). It is now possible to express equations (2) in terms of t and u, alone. Indeed,

$$\frac{\partial n}{\partial R} = \frac{\partial t}{\partial R}\cot\phi$$

$$\frac{\partial n}{\partial \alpha} = \frac{\partial t}{\partial \alpha}\cot\phi$$

$$\frac{\partial m}{\partial \alpha} = \frac{\partial t}{\partial \alpha}(\cot\phi + 2\tan\phi)$$

Introducing these values in (2), the following system is obtained:

$$\left.\begin{aligned} &\frac{\partial t}{\partial R}\cot\phi + \frac{\partial u}{\partial R} - \frac{1}{R}\frac{\partial t}{\partial \alpha} - \frac{2t\tan\phi}{R} - \gamma\cos\alpha = 0 \\ &\frac{1}{R}\frac{\partial t}{\partial \alpha}(\cot\phi + 2\tan\phi) + \frac{1}{R}\frac{\partial u}{\partial \alpha} - \frac{\partial t}{\partial R} - \frac{2t}{R} + \gamma\sin\alpha = 0 \end{aligned}\right\} \quad \ldots\ldots(3)$$

$\dfrac{\partial t}{\partial R}$ can be eliminated from equations (3), thus obtaining

$$\frac{\partial t}{\partial \alpha} = 2t\tan\phi - \gamma R\sin\phi\sin(\alpha - \phi) - \frac{\partial u}{\partial \alpha}\sin\phi\cos\phi - R\frac{\partial u}{\partial R}\sin^2\phi \quad \ldots(4)$$

which is the required equation.

Simple Application of Equation (4)

Consider a soil submerged to the top, so that $u = \gamma_0 h$, where γ_0 is the unit weight of water and h is the depth. Then,

$$\frac{\partial u}{\partial \alpha} = -\gamma_0 R\sin\alpha$$

$$\frac{\partial u}{\partial R} = \gamma_0\cos\alpha$$

Therefore equation (4) reduces in this case to

$$\frac{\partial t}{\partial \alpha} = 2t\tan\phi - \gamma R\sin\phi\sin(\alpha - \phi) + \gamma_0 R\sin\phi\sin(\alpha - \phi)$$

$$= 2t\tan\phi - (\gamma - \gamma_0)R\sin\phi\sin(\alpha - \phi)$$

That is, the sliding line conforms to Kötter's equation, provided the buoyant unit weight of the material is utilized instead of the total unit weight.

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FRENET FORMULAS FOR CURVES IN UNITARY SPACE

By N. Coburn

I. Introduction

We shall develop a set of Frenet formulas for any curve X_1 whose equations are functions of a real parameter (t) and which lies imbedded in a unitary space of n-dimensions K_n. These formulas are developed by analogy with the Frenet formulas for curves V_1 imbedded in Riemannian space of n-dimensions V_n (1). However, the X_1 in K_n possesses $(3n - 2)$ non-zero curvatures, of which $(2n - 1)$ are independent. In Riemannian space, the curvature matrix of V_1 in V_n is skew-symmetric. Somewhat analogously, the curvature matrix of X_1 in K_n possesses Hermitian symmetry. Finally, corresponding to the similar theorem (2) in Riemannian space, we prove: a complex curve X_1 in a unitary space K_n is uniquely determined when an initial point, an initial orientation of the n-nuple of normals, and the $(2n - 1)$ independent curvatures are given.

The paper concludes with a study of geodesic (3) X_1 in a unitary K_n with semi-symmetric connection. In such spaces, it is shown that a geodesic X_1 is characterized by: (1) the (0, 0) curvature vanishes; (2) the (0, 1) curvature is the negative of the projection of the "semi-symmetric" vector on the first normal; (3) the "semi-symmetric" vector of K_n lies in the plane of the tangent vector and the first normal of X_1 (that is, the osculating U_2).

II. Notation (4)

Consider a real space of $2n$-dimensions X_{2n} whose coordinates are given by the real variables

$$x^\lambda, y^\lambda, \lambda, \mu = 1 \cdots n. \tag{2.1}$$

Into this X_{2n}, we introduce the complex coordinates given by

$$\begin{aligned} \xi^\lambda &= x^\lambda + iy^\lambda, \\ \xi^{\lambda^*} &= x^\lambda - iy^\lambda. \end{aligned} \qquad \lambda, \mu = 1 \cdots n \tag{2.2}$$

Since the Jacobian of this transformation $(-2i)$ does not vanish over X_{2n}, the ξ^λ, ξ^{λ^*} constitute a set of $2n$ independent variables which map the X_{2n}. In view of the fact that ξ^{λ^*} are complex conjugate to ξ^λ, we can determine the points of X_{2n} by assigning complex numbers to merely ξ^λ. Hence, we say that the ξ^λ determine "points" which build a *complex* space of n-dimensions X_n (the above real topological X_{2n}). Let us denote partial derivatives by

$$\partial_\mu = \partial/\partial\xi^\mu, \qquad \partial_{\mu^*} = \partial/\partial\xi^{\mu^*}, \tag{2.3}$$

and let $\phi(\xi^\lambda, \xi^{\lambda^*})$ be an analytic function of the variables ξ^λ, ξ^{λ^*}. Then by the composite function theorem, we obtain

$$\partial\phi/\partial x^\mu = \partial_\mu\phi + \partial_{\mu^*}\phi, \qquad \lambda, \mu = 1, \cdots n, \tag{2.4}$$

$$\partial\phi/\partial y^\mu = i\partial_\mu\phi - i\partial_{\mu^*}\phi. \tag{2.5}$$

Solving for $\partial_\mu \phi$, $\partial_{\mu^*} \phi$, we find

$$\partial_\mu \phi = 1/2(\partial\phi/\partial x^\mu - i\partial\phi/\partial y^\mu), \tag{2.6}$$

$$\partial_{\mu^*} \phi = 1/2(\partial\phi/\partial x^\mu + i\partial\phi/\partial y^\mu). \tag{2.7}$$

If the function $\psi(\xi^\lambda)$,

$$\psi(\xi^\lambda) = u(x^\lambda, y^\lambda) + iv(x^\lambda, y^\lambda), \qquad \lambda, \mu = 1 \cdots n, \tag{2.8}$$

is analytic in the sense of Cauchy-Riemann, then

$$\partial u/\partial x^\lambda - \partial v/\partial y^\lambda = 0, \qquad \partial v/\partial x^\lambda + \partial u/\partial y^\lambda = 0. \tag{2.9}$$

Expanding the right hand side of (2.7), we find that (2.9) is equivalent to

$$\partial_{\lambda^*} \psi = 0. \tag{2.10}$$

From our point of view, this equation follows directly from the fact that ξ^λ, ξ^{λ^*} are independent variables. Hence equation (2.9) serves merely to interpret the equation (2.10) from the point of view of complex variable. If ψ^* is the complex conjugate to ψ,

$$\psi^* = u - iv, \tag{2.11}$$

then it is easily shown that (2.9) is equivalent to the so-called conjugate equation

$$\partial_\lambda \psi^* = 0. \tag{2.12}$$

We seek to generalize the idea involved in (2.12). Corresponding to any function $\phi(\xi^\lambda, \xi^{\lambda^*})$, let $\phi^*(\xi^{\lambda^*}, \xi^\lambda)$ denote the function obtained by replacing (i) by $(-i)$ when the function are expanded in terms of the real variables x^λ, y^λ. From (2.2), we see that the variables ξ^λ are then replaced by ξ^{λ^*} and conversely. We call this function ϕ^*, the conjugate of ϕ. In the future, we shall indicate the validity of the conjugate by the abbreviation "conj." It is to be noted on the formal side, that in passing to the conjugate, all functions and indices will be starred. The star of a starred quantity removes the original star.

Consider the allowable analytic coordinate transformation with non-vanishing Jacobian

$$\xi^\lambda = \xi^\lambda(\xi^{\lambda'}), \qquad \xi^{\lambda'} = x^{\lambda'} + iy^{\lambda'}, \qquad \lambda', \mu' = 1 \cdots n. \tag{2.13}$$

The corresponding conjugate equations are

$$\xi^{\lambda^*} = \xi^{\lambda^*}(\xi^{\lambda^{*'}}), \qquad \xi^{\lambda^{*'}} = x^{\lambda'} - iy^{\lambda'}. \tag{2.14}$$

We now introduce the unit affinor whose intermediary (5) components are

$$A^\lambda_{\mu'} = \partial\xi^\lambda/\partial\xi^{\mu'}, \qquad A^{\lambda^*}_{\mu^{*'}} = \partial\xi^{\lambda^*}/\partial\xi^{\mu^{*'}}. \tag{2.15}$$

A vector $v^\lambda(\xi^\lambda, \xi^{\lambda^*})$ is said to be of the *first type* if

$$v^{\lambda'} = A^{\lambda'}_\lambda v^\lambda \tag{2.16}$$

Associated with each such vector is its conjugate, or vector of the *second type* $v^{\lambda^\bullet}$ with transfomation law

$$v^{\lambda^{\bullet\prime}} = A^{\lambda^{\bullet\prime}}_{\lambda^\bullet} v^{\lambda^\bullet} \tag{2.17}$$

The theory can be extended to affinors of any mixed valence.

Let us introduce a connection in X_n, by means of the n^3 quantities $\Gamma^\lambda_{\mu\alpha}$ which are functions of position. Then, we define the covariant differential of a contravariant vector by

$$\delta v^\lambda = dv^\lambda + \Gamma^\lambda_{\mu\alpha} v^\alpha \, d\xi^\mu, \text{ conj.} \tag{2.18}$$

The $\Gamma^\lambda_{\mu\alpha}$ transform in a well known manner (6) under (2.13). We now write for a covariant vector

$$\delta w_\lambda = dw_\lambda - \Gamma^\alpha_{\mu\lambda} w_\alpha \, d\xi^\mu, \text{ conj.} \tag{2.19}$$

By expanding the ordinary differential of a vector, we obtain

$$dv^\lambda = d\xi^\mu \partial_\mu v^\lambda + d\xi^{\mu^\bullet} \partial_{\mu^\bullet} v^\lambda, \text{ conj.} \tag{2.20}$$

If we define the covariant derivative of v^λ, w_λ by means of

$$\nabla_\mu v^\lambda = \partial_\mu v^\lambda + \Gamma^\lambda_{\mu\alpha} v^\alpha, \qquad \nabla_\mu w_\lambda = \partial_\mu w_\lambda - \Gamma^\alpha_{\mu\lambda} w_\alpha, \tag{2.21}$$

$$\nabla_{\mu^\bullet} v^\lambda = \partial_{\mu^\bullet} v^\lambda, \qquad \nabla_{\mu^\bullet} w_\lambda = \partial_{\mu^\bullet} w_\lambda, \text{ conj.}, \tag{2.22}$$

then

$$\delta v^\lambda = d\xi^\mu \nabla_\mu v^\lambda + d\xi^{\mu^\bullet} \nabla_{\mu^\bullet} v^\lambda, \text{ conj.}, \tag{2.23}$$

$$\delta w_\lambda = d\xi^\mu \nabla_\mu w_\lambda + d\xi^{\mu^\bullet} \nabla_{\mu^\bullet} w_\lambda, \text{ conj.} \tag{2.24}$$

An Hermitian X_n, with covariant derivative defined by (2.21), (2.22) is denoted by K_n.

We now introduce an Hermitian tensor with Hermitian symmetry

$$a_{\lambda\mu^\bullet} = [(a_{\lambda\mu^\bullet})^*]' = a_{\mu^\bullet\lambda}, \tag{2.25}$$

the $(')$ indicating the transpose matrix. If we condition $a_{\lambda\mu^\bullet}$ by requiring that

$$\delta a_{\lambda\mu^\bullet} = 0 = (\partial_\rho g_{\lambda\mu^\bullet} - \Gamma^\sigma_{\rho\lambda} a_{\sigma\mu^\bullet}) \, d\xi^\rho + (\partial_{\rho^\bullet} a_{\lambda\mu^\bullet} - \Gamma^{\sigma^\bullet}_{\rho^\bullet\mu^\bullet} a_{\lambda\sigma^\bullet}) \, d\xi^{\rho^\bullet}, \tag{2.26}$$

then the space K_n is said to be a unitary K_n. For such a space from (2.26), we can prove (7)

$$\nabla_\rho a_{\lambda\mu^\bullet} = \partial_\rho a_{\lambda\mu^\bullet} - \Gamma^\sigma_{\rho\lambda} a_{\sigma\mu^\bullet} = 0, \text{ conj.}, \tag{2.27}$$

$$\nabla_{\rho^\bullet} a_{\lambda\mu^\bullet} = \partial_{\rho^\bullet} a_{\lambda\mu^\bullet} - \Gamma^{\sigma^\bullet}_{\rho^\bullet\mu^\bullet} a_{\lambda\sigma^\bullet} = 0, \text{ conj.} \tag{2.28}$$

The $a_{\lambda\mu^\bullet}$ is now a fundamental tensor and can be used to raise and lower indices through the ∇ operator. If we define the contravariant fundamental tensor $a^{\sigma^\bullet\lambda}$ by

$$a^{\sigma^\bullet\lambda} a_{\mu\sigma^\bullet} = A^\lambda_\mu, \text{ conj.}, \tag{2.29}$$

then (2.27), (2.28) may be solved for the connection

$$\Gamma^{\alpha}_{\mu\lambda} = (\partial_\mu a_{\lambda\sigma^*})a^{\sigma^*\alpha} = (\partial_\mu a_{\sigma^*\lambda})a^{\alpha\sigma^*}, \tag{2.30}$$

$$\Gamma^{\alpha^*}_{\mu^*\lambda^*} = (\partial_{\mu^*} a_{\sigma\lambda^*})a^{\alpha^*\sigma} = (\partial_{\mu^*} a_{\lambda\sigma^*})a^{\sigma\alpha^*}. \tag{2.31}$$

Finally, we introduce the torsion affinor

$$S^{\cdots\alpha}_{\mu\lambda} = \tfrac{1}{2}(\Gamma^{\alpha}_{\mu\lambda} - \Gamma^{\alpha}_{\lambda\mu}) = \Gamma^{\alpha}_{[\mu\lambda]}. \tag{2.32}$$

The sign [ ] means that the antisymmetric product of the enclosed indices is to be formed; the sign ( ) means that the symmetric product of the enclosed indices is to be formed. When the torsion affinor can be written as

$$S^{\cdots\alpha}_{\mu\lambda} = A^{\alpha}_{[\mu}p_{\lambda]}, \quad \text{conj.}, \tag{2.33}$$

then the space K_n is said to have a semi-symmetric connection.

III. Frenet Formulas for X_1 in K_n.

Consider the curve X_1 defined in K_n by means of the equations

$$\xi^{\lambda} = \xi^{\lambda}(t), \qquad \xi^{\lambda^*} = \xi^{\lambda^*}(t), \tag{3.1}$$

where (t) is a real parameter and ξ^{λ}, ξ^{λ^*} are conjugate functions. Evidently, such equations determine a curve X_1 in the real topological X_{2n}. We define the element of arc length along this X_1 by

$$ds = (a_{\lambda\mu^*}\, d_t\xi^{\lambda}\, d_t\xi^{\mu^*})^{1/2}\, dt, \text{ conj.}, \qquad (d_t\xi^{\lambda} = d\xi^{\lambda}/dt). \tag{3.2}$$

By substituting (3.1) into (3.2) and integrating the latter equation, we obtain (s), the arc length parameter, in terms of (t). It is to be noted that this parameter exists only because of the fact that (t) is real. By use of (2.25) and the conjugate of (3.2), we see that (ds) and hence (s) are real.

We now proceed to the Frenet formulas for X_1 in K_n. Our deduction proceeds similarly to that in Riemannian space. The differences in the resulting formulas are due to the differences in space structure. We shall denote the unit unitary tangent vector to X_1 by

$$\underset{0}{u}{}^{\lambda} = d_s\xi^{\lambda}, \quad \text{conj.} \tag{3.3}$$

If we restrict our discussion to unitary spaces K_n in which $a_{\lambda\mu^*}$ is positive definite (8), then

$$\underset{0}{u}{}^{\lambda}\,\underset{0}{u}{}_{\lambda} = 1, \quad \text{conj.} \tag{3.4}$$

We next introduce the first unit normal vector u^{λ} (subindex 1)

$$\underset{1}{u}{}^{\lambda}\,\underset{1}{u}{}_{\lambda} = 1, \qquad \underset{1}{u}{}^{\lambda}\,\underset{0}{u}{}_{\lambda} = 0, \quad \text{conj.}, \tag{3.5}$$

which is defined by the equation

$$\delta_s \underset{0}{u}{}^{\lambda} = k\underset{00}{u}{}^{\lambda}\underset{0}{} + k\underset{01}{u}{}^{\lambda}\underset{1}{}, \qquad (\delta_s = \delta/ds), \text{ conj.} \tag{3.6}$$

The coefficients k (subindex 00, 01) are the (0, 0) and (0, 1) curvatures, respectively. From (3.4), we obtain

$$u_{\lambda}\,\delta_s u^{\lambda}_{0\ 0} + u^{\lambda}_{0}\,\delta_s u_{\lambda\ 0} = 0, \quad \text{conj.} \tag{3.7}$$

By use of the fundamental tensor, we find

$$u^{\lambda}_{0}\,\delta_s u_{\lambda\ 0} = u^{\lambda}_{0}\,\delta_s(a_{\lambda\mu^*}\,u^{\mu^*}_{0}) = u_{\mu^*}\,\delta_s u^{\mu^*}_{0\ 0}. \tag{3.8}$$

Taking the conjugate of (3.6) and transvecting with u_{μ^*} (subindex 0), we obtain after simplifying with the aid of (3.5)

$$u_{\mu^*}\,\delta_s u^{\mu^*}_{0\ 0} = k^*_{00}, \quad \text{conj.} \tag{3.9}$$

From (3.9), (3.8), we see that (3.7) becomes

$$k_{00} + k^*_{00} = 0, \quad \text{conj.} \tag{3.10}$$

Hence k (subindex 00) is a pure imaginary. We next introduce the second unit normal vector u^{λ} (subindex 2)

$$u^{\lambda}_{2}\,u_{\lambda\ 2} = 1, \qquad u^{\lambda}_{2}\,u_{\lambda\ 1} = 0, \qquad u^{\lambda}_{2}\,u_{\lambda\ 0} = 0, \quad \text{conj.}, \tag{3.11}$$

which is defined by the equation

$$\delta_s u^{\lambda}_{1} = k_{10}\,u^{\lambda}_{0} + k_{11}\,u^{\lambda}_{1} + k_{12}\,u^{\lambda}_{2}, \quad \text{conj.} \tag{3.12}$$

The coefficients k (subindex 10, 11, 12) are three new curvatures. By differentiation of (3.5), we obtain

$$u_{\lambda}\,\delta_s u^{\lambda}_{0\ 1} + u^{\lambda}_{1}\,\delta_s u_{\lambda\ 0} = 0, \quad \text{conj.} \tag{3.13}$$

The first term in (3.13) can be evaluated by transvecting (3.12) with u_{λ} (subindex 0). Proceeding as in (3.8), (3.9), the second term may be evaluated. The result of these calculations is

$$k_{10} + k^*_{01} = 0, \quad \text{conj.} \tag{3.14}$$

By taking the covariant derivative with respect to (s) of

$$u^{\lambda}_{1}\,u_{\lambda\ 1} = 1, \quad \text{conj.} \tag{3.15}$$

and transvecting (3.12) with u_{λ} (subindex 1), we obtain

$$k_{11} + k^*_{11} = 0, \tag{3.16}$$

Again, we introduce a new unit normal vector- the third unit normal vector u^{λ} (subindex 3)

$$u^{\lambda}_{3}\,u_{\lambda\ j} = \delta_{3j}, \quad j = 0, 1, 2, 3, \quad \text{conj.} \tag{3.17}$$

which is defined by (δ is the Kronecker symbol)

$$\delta_2 u^\lambda = k_{20}\, u^\lambda_0 + k_{21}\, u^\lambda_1 + k_{22}\, u^\lambda_2 + k_{23}\, u^\lambda_3, \quad \text{conj.} \tag{3.18}$$

We now show the general scheme for finding relations of the form (3.14), (3.16). From the relations

$$u_{j\lambda} u^\lambda_2 = \delta_{j2}, \quad j = 0, 1, 2, 3, \quad \text{conj.}, \tag{3.19}$$

we find by covariant differentiation and proper use of the fundamental tensor (see 3.8)

$$u_{j\lambda} \delta_2 u^\lambda_2 + u_{j\lambda^*} \delta_2 u^{\lambda^*}_j = 0, \quad j = 0, 1, 2, 3, \quad \text{conj.} \tag{3.20}$$

For all (j), the first term in (3.20) can be computed with the aid of (3.18). To compute the second term of (3.20), for $j = 0$, use (3.6); for $j = 1$, use (3.12); for $j = 2$, use (3.18). Our results are

$$k_{2j} + k^*_{j2} = 0, \quad j = 0, 1, 2, \quad \text{conj.} \tag{3.21}$$

In particular, it follows from (3.21), (3.6), that

$$k_{20} = 0. \tag{3.22}$$

By an easy induction, one can verify that the Frenet formulas for an X_1 in K_n are

$$\begin{aligned}
\delta_0 u^\lambda_0 &= k_{00}\, u^\lambda_0 + k_{01}\, u^\lambda_1, && \text{conj.}\\
&\vdots && \vdots\\
\delta_1 u^\lambda_1 &= k_{10}\, u^\lambda_0 + k_{11}\, u^\lambda_1 + k_{12}\, u^\lambda_2,\\
\delta_2 u^\lambda_2 &= \quad k_{21}\, u^\lambda_1 + k_{22}\, u^\lambda_2 + k_{23}\, u^\lambda_3,\\
&\vdots\\
\delta_j u^\lambda_j &= \quad k_{j\,j-1}\, u^\lambda_{j-1} + k_{jj}\, u^\lambda_j + k_{j\,j+1}\, u^\lambda_{j+1},\\
&\vdots\\
\delta_{n-1} u^\lambda_{n-1} &= \quad k_{n-1\,n-2}\, u^\lambda_{n-2} + k_{n-1\,n-1}\, u^\lambda_{n-1}
\end{aligned} \tag{3.23}$$

where the $3(n-2) + 4 = 3n - 2$ curvatures k are related by

$$k_{jl} + k^*_{lj} = 0, \quad j, l = 0, 1, \cdots n-1, \quad \text{conj.} \tag{3.24}$$

Hence, we have the theorem

THEOREM 1: The formulas (3.23), (3.24) constitute the Frenet formulas for X_1 in K_n.

If we substitute (3.24) into (3.23) and write the matrix of the curvature quantities k, we obtain

$$
(3.25)\qquad
\begin{bmatrix}
k_{00} & k_{01} & 0 & 0 & 0 & \cdots\cdots\cdots\cdots \\
-k^*_{01} & k_{11} & k_{12} & 0 & 0 & \cdots\cdots\cdots\cdots \\
0 & -k^*_{12} & k_{22} & k_{23} & 0 & \cdots\cdots\cdots\cdots \\
0 & 0 & \cdot & \cdot & \cdot & 0\cdots\cdots\cdots\cdots \\
\vdots & & & & & \\
\cdots\cdots\cdots\cdots\cdots & & & & -k^*_{n-2\,n-1} & k_{n-1\,n-1}
\end{bmatrix}
$$

where k (subindex jj) are pure imaginaries. If we place

$$
(3.26)\qquad k_{jl} = i\,k'_{jl}, \quad \text{conj.}
$$

then (3.24) becomes

$$
(3.27)\qquad k'_{jl} = k^{*\prime}_{lj}, \quad \text{conj.}
$$

That is, the curvature quantities k' are Hermitian symmetric. Likewise, the curvature matrix of the k' quantities, which corresponds to (3.25), possesses Hermitian symmetry. The curvature quantities k' can be introduced into (3.23), by dividing both sides of each equation by (i). The set (3.23), then, becomes,

$$
(3.28)\qquad
\begin{aligned}
i^{-1}\delta_s\, u^\lambda_0 &= k'_{00}\, u^\lambda_0 + k'_{01}\, u^\lambda_1, \quad \text{conj.}, \\
&\vdots \qquad\qquad \vdots \\
i^{-1}\delta_s\, u^\lambda_{n-1} &= \cdot \qquad\qquad k'_{n-1\,n-2}\, u^\lambda_{n-2} + k'_{n-1\,n-1}\, u^\lambda_{n-1}
\end{aligned}
$$

From the form of (3.25) or the similar matrix in k', we see that the $(3n-2)$ curvatures contain $n + (n-1) = 2n-1$ independent curvatures. Hence we have the theorem

THEOREM 2: By transforming the Frenet formulas (3.23) into the form (3.24), the resulting curvature matrix, which is composed of $(2n-1)$ independent curvatures, possesses Hermitian symmetry.

We close this section by proving the theorem

THEOREM 3: A complex curve X_1 in a unitary space K_n is uniquely determined when the following are known: (1) an initial point; (2) an initial orientation of the n-nuple of normals; (3) $(2n-1)$ independent curvatures as functions of the arc length parameter (s).

This theorem and its proof are completely analogous to that for V_1 in V_n (2). As in Riemannian space, the critical question is whether the relations

$$
(3.29)\qquad \delta_s(u^\lambda_j\, u_{\lambda l}) = 0, \quad j, l = 0, 1 \cdots n-1, \quad \text{conj.}
$$

are valid. By use of the equations (3.23), (3.24), we shall show that (3.29) is valid. Upon expanding (3.29), we obtain

$$u_{\lambda}\,\delta_s\,u^{\lambda} + u_{\lambda^*}\,\delta_s\,u^{\lambda^*} = 0, \quad \text{conj.} \tag{3.30}$$

Substituting (3.23) into (3.30), we find

$$u_{\lambda}\left(k\,u^{\lambda} + k\,u^{\lambda} + k\,u^{\lambda}\right) + u_{\lambda^*}\left(k^*\,u^{\lambda^*} + k^*\,u^{\lambda^*} + k^*\,u^{\lambda^*}\right) = 0. \tag{3.31}$$

By simplifying (3.31), we obtain

$$k\,\delta + k\,\delta + k\,\delta + k^*\,\delta + k^*\,\delta + k^*\,\delta = 0. \tag{3.32}$$

We analyze (3.32) by cases.

Case 1: $l = j - 1$, hence $j = l + 1$. Hence (3.32) becomes

$$k + k^* = 0. \tag{3.33}$$

Case 2: $l = j$. Equation (3.32) becomes

$$k + k^* = 0. \tag{3.34}$$

Case 3: $l = j + 1$, hence $j = l - 1$. Thus (3.32) becomes

$$k + k^* = 0. \tag{3.35}$$

All other possibilities are ruled out by the occurrence of the Kronecker symbols in (3.32). Furthermore, (3.33), (3.34), (3.35) are satisfied in virtue of (3.24). Hence (3.29) is valid and our theorem is demonstrated.

IV. Geodesics of a Unitary K_n with Semi-symmetric Connection.

The torsion affinor of a unitary K_n with semi-symmetric connection is given by (see 2.33)

$$S_{\nu\lambda}^{\cdot\cdot\kappa} = A^{\kappa}_{[\nu}p_{\lambda]}\,, \text{ conj.} \tag{4.1}$$

It can be shown that the geodesic X_1 which depend on a real parameter (t) are the solutions of (3)

$$\delta_s\,u^{\alpha} + 2a_{\mu\kappa^*}\,a^{\lambda^*\alpha}\,S_{\nu^*\lambda^*}^{\cdot\cdot\kappa^*}\,u^{\nu^*}\,u^{\mu} = 0, \quad \text{conj.} \tag{4.2}$$

With the aid of (4.1), the second term in (4.2) can be reduced to

$$p^{\alpha} - u^{\alpha}(p^{\nu}u_{\nu}). \tag{4.3}$$

Substituting (3.6) and (4.3) into (4.2), the latter equation becomes

$$k\,u^{\alpha} + k\,u^{\alpha} + p^{\alpha} - u^{\alpha}(p^{\nu}u_{\nu}) = 0, \quad \text{conj.} \tag{4.4}$$

By decomposing p^a with respect to the n-nuple of u^a vectors, we obtain

$$p^a = \sum_{j=0}^{n-1} (p^r u_r)u^a, \quad \text{conj.} \tag{4.5}$$

Placing (4.5) into (4.4), the latter becomes

$$k_{00}\, u^a + k_{01}\, u^a + \sum_{j=1}^{n-1} (p^r u_r)u^a = 0, \quad \text{conj.} \tag{4.6}$$

Hence, we deduce the relations

$$k_{00} = 0, \quad p^r u_r = -k_{01}, \quad \text{conj.} \tag{4.7}$$

$$p^r u_r = 0, \quad j = 2 \cdots n-1, \quad \text{conj.} \tag{4.8}$$

Thus, we have the theorem

THEOREM 4: The geodesic X_1 of a unitary K_n with semi-symmetric connection are characterized by: (1) the (0, 0) curvature vanishes; (2) the (0, 1) curvature is the negative of the projection of the vector p^a on the first normal to X_1; (3) the osculating U_2 of X_1 contains p^a.

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NOTE ON THE INTEGRO-DIFFERENTIAL EQUATION OF A PROBLEM IN THE THEORY OF PLANE STRESS

By F. B. Hildebrand

In a recent paper[1] E. Reissner considered the problem of determining the distribution of stress in a semi-infinite elastic sheet to which a stiffener of length L and cross-sectional area A is attached. The direction of the stiffener was taken to be normal to the edge of the sheet, the cross-section of the stiffener being symmetrical about the middle surface of the sheet, and a concentrated force F was assumed to be acting at one end of the stiffener. It was shown in this case that the stiffener normal stress σ, at a distance x from the point of load application, is determined by the equation

$$\sigma(x) + \lambda A(L)\sigma(L)\left\{\frac{1}{L-x} + K(L,x)\right\} = \lambda \mathbf{P}\int_0^L [A(\xi)\sigma(\xi)]'\left\{\frac{1}{\xi - x} + K(\xi, x)\right\} d\xi, \qquad (1)$$

$$(0 < x < L,\ x \to 0_+,\ x \to L_-)$$

with

$$K(\xi, x) = \frac{B_1 x\xi}{(\xi + x)^3} + \frac{B_2 x + B_3 \xi}{(\xi + x)^2} + \frac{B_4}{\xi + x},$$

where λ and the B_i are certain constants depending upon the relative dimensions and elastic properties of the stiffener and sheet. Here and in the sequel the integrals are defined in the sense of Riemann, the symbol $\mathbf{P}$ being used, when necessary, to indicate that the Cauchy principal value of an integral is to be taken.

Reissner conjectured that a solution of this equation for which the axial stiffener force $A(x)\sigma(x)$ is continuous in the closed interval $(0, L)$ and the shear stress $[A(x)\sigma(x)]'$ between stiffener and sheet is continuous in the corresponding open interval would necessarily have $A(L)\sigma(L) = 0$, so that no concentrated force is introduced into the sheet at the unloaded end $(x = L)$ of the stiffener. Thus the solution of (1) should satisfy the two conditions

$$A(0)\sigma(0) = F, \qquad A(L)\sigma(L) = 0, \qquad (2)$$

leading to the abbreviated equation

$$\sigma(x) = \lambda \mathbf{P}\int_0^L [A(\xi)\sigma(\xi)]'\left\{\frac{1}{\xi - x} + K(\xi, x)\right\} d\xi, \qquad (1a)$$

and the boundary conditions (2).

[1] E. Reissner, *Note on the Problem of the Distribution of Stress in a Thin Stiffened Elastic Sheet*, Proc. Nat. Acad. Sci., Vol. 26, No. 4, pp. 300–305, April, 1940.

The purpose of the present note is to establish the fact that the added stipulation that $(L - x)[A(x)\sigma(x)]'$ be absolutely integrable over some interval $(L - \delta, L)$ is sufficient to show the necessity of the condition $A(L)\sigma(L) = 0$. Since all integrable functions $[A(x)\sigma(x)]'$ for which $|A[(x)\sigma(x)]'| < (L - x)^{-2+\epsilon}$ near $x = L$ satisfy the former condition, it would seem that the limitation is not essentially restrictive.

To establish the desired result, we first transform (1) into a more convenient form by introducing into (1) the identity

$$A(L)\sigma(L)\frac{1}{L-x} = \int_0^L [A(\xi)\sigma(\xi)]'\frac{d\xi}{L-x} + A(0)\sigma(0)\frac{1}{L-x} \tag{3}$$

and multiplying both sides of the resultant equation by $(L - x)$. This gives the equivalent relationship

$$\lambda A(0)\sigma(0) + (L-x)\{\sigma(x) + \lambda K(L, x)\}$$
$$= \lambda \mathbf{P}\int_0^L [A(\xi)\sigma(\xi)]'\frac{L-\xi}{\xi-x}\,d\xi + (L-x)\lambda\int_0^L [A(\xi)\sigma(\xi)]' K(\xi, x)\,d\xi. \tag{4}$$

An integration by parts gives

$$\int_0^L [A(\xi)\sigma(\xi)]' K(\xi, x)\,d\xi$$
$$= \left[K(t, x)\int_0^t [A(\xi)\sigma(\xi)]'\,d\xi\right]_{t=0}^{t=L} - \int_0^L \left\{\int_0^t [A(\xi)\sigma(\xi)]'\,d\xi\right\}\frac{\partial K(t, x)}{\partial t}\,dt.$$

Since $\sigma(x)$, $A(x)\sigma(x)$, $K(\xi, x)$, and $\dfrac{\partial K(\xi, x)}{\partial \xi}$ are all bounded when $x > 0$, it follows that this integral, as well as the expression in brackets on the left hand side of (4), is bounded for positive x. Hence, as x approaches L from the left $(x \to L_-)$, we obtain

$$A(0)\sigma(0) = \lim_{x\to L_-} \mathbf{P}\int_0^L [A(\xi)\sigma(\xi)]'\frac{L-\xi}{\xi-x}\,d\xi, \tag{5}$$

or, by the definition of a Cauchy principal value,

$$A(0)\sigma(0) = \lim_{x\to L_-}\lim_{\epsilon\to 0_+}\left\{\int_0^{x-\epsilon} + \int_{x+\epsilon}^L\right\}[A(\xi)\sigma(\xi)]'\frac{L-\xi}{\xi-x}\,d\xi. \tag{5a}$$

If a rather complicated inversion of limits is legitimate, this is equivalent to the condition

$$A(0)\sigma(0) = -[A(\xi)\sigma(\xi)]_0^L, \qquad A(L)\sigma(L) = 0, \tag{6}$$

which is to be established.

In case $(L - x)[A(x)\sigma(x)]'$ is absolutely integrable over $(L - \delta, L)$, for some

positive δ, the transition from (5) to (6), which depends only upon the truth of the statement.

$$\lim_{x\to L-} \mathbf{P}\int_0^L [A(\xi)\sigma(\xi)]' \frac{L-\xi}{\xi - x}\, d\xi = \int_0^L [A(\xi)\sigma(\xi)]'\, d\xi, \tag{7}$$

is justified, after an obvious transformation, by the following:

THEOREM. *Suppose that* (i) $f'(x)$ *is integrable over the interval* $(0, 1)$, (ii) $f(x) = \int_0^x f'(\xi)\, d\xi - f(0)$ *is continuous in the closed interval, and* (iii) $xf'(x)$ *is absolutely integrable* (and hence integrable in the sense of Lebesgue) *over an interval having* $x = 0$ *as left-hand end point. Then*

$$\lim_{x\to 0+} \mathbf{P}\int_0^1 \xi f'(\xi) \frac{d\xi}{\xi - x} = \int_0^1 f'(\xi)\, d\xi.$$

PROOF. Setting $g(x) = xf'(x)$, it follows that, for $0 < x < 1$,

$$\begin{aligned}
\mathbf{P}\int_0^1 g(\xi)\frac{d\xi}{\xi - x} &= \lim_{\epsilon\to 0+}\left\{\int_0^{x-\epsilon} + \int_{x+\epsilon}^1\right\} \\
&= \lim_{\epsilon\to 0+}\int_\epsilon^x \frac{g(x+t) - g(x-t)}{t}\, dt + \int_x^{1-x} g(x+t)\frac{dt}{t} \\
&= \lim_{\epsilon\to 0+}\int_\epsilon^x \frac{g(x+t) - g(x-t)}{t}\, dt + \int_{2x}^1 f'(\xi)\, d\xi + x\int_{2x}^1 f'(\xi)\frac{d\xi}{\xi - x}.
\end{aligned}$$

By a well-known theorem of Plessner,[2] the integral $\int_\epsilon^x \frac{g(x+t) - g(x-t)}{t}\, dt$ converges as $\epsilon \to 0_+$ for almost all x if condition (iii) is satisfied,[3] and hence approaches zero with x. Also, integrating by parts, and applying condition (ii), we have

$$\begin{aligned}
\int_{2x}^1 f'(\xi)\frac{d\xi}{\xi - x} &= \left[\frac{1}{t - x}\int_{2x}^t f'(\xi)\, d\xi\right]_{t=2x}^{t=1} + \int_{2x}^1 \left\{\int_{2x}^t f'(\xi)\, d\xi\right\}\frac{dt}{(t-x)^2} \\
&= 0(1) + \int_{2x}^{x+\sqrt{x}} \left\{\int_{2x}^t f'(\xi)\, d\xi\right\}\frac{dt}{(t-x)^2} + \int_{x+\sqrt{x}}^1 \left\{\int_{2x}^t f'(\xi)\, d\xi\right\}\frac{dt}{(t-x)^2}
\end{aligned}$$

[2] A. Plessner, *Zur Theorie der konjugierte trigonometrischen Reihen*, Mitteilungen des Math. Seminar d. Univ. Giessen 10 (1923), page 18.

[3] Plessner's theorem states that $\lim_{\epsilon\to 0+}\int_\epsilon^x |g(x+t) - g(x-t)| \cot\frac{t}{2}\, dt$ *exists in the sense of Lebesgue for almost all* x *if* $g(x)$ *is summable.* This implies the existence of $\lim_{\epsilon\to 0+}\int_\epsilon^x$ for almost all x. The theorem is also true if $\cot\frac{t}{2}$ be replaced by $\frac{1}{t}$ since $\cot\frac{t}{2} = \frac{2}{t} + 0(t)$, $t \to 0$. Because of the absolute integrability of $g(x)$, the Riemann and Lebesgue integrals are identical.

$$= 0(1) - \left\{\int_{2x}^{\xi_1} f'(\xi)\,d\xi\right\}\left[\frac{1}{t-x}\right]_{t=2x}^{t=x+\sqrt{x}} - \left\{\int_{2x}^{\xi_2} f'(\xi)\,d\xi\right\}\left[\frac{1}{t-x}\right]_{t=x+\sqrt{x}}^{t=1},$$

$$(2x \leqq \xi_1 \leqq x + \sqrt{x},\ x + \sqrt{x} \leqq \xi_2 \leqq 1)$$

$$= 0(1) + o\left(\frac{1}{x}\right) + 0\left(\frac{1}{\sqrt{x}}\right), \qquad x \to 0_+,$$

so that $x\int_{2x}^{1} f'(\xi)\frac{d\xi}{\xi - x} = o(1)$, $x \to 0_+$. It follows that

$$\lim_{x\to 0_+} \int_0^1 g(\xi)\frac{d\xi}{\xi - x} = \int_0^1 f'(\xi)\,d\xi,$$

establishing the theorem.

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TRANSMISSION THEORY OF A CYLINDRICAL HOLLOW TUBE GUIDE

By Chang-Pen Hsü

Introduction

As the title of the paper indicates, the problem consists of a mathematical solution of Maxwell's field equations in cylindrical coordinates under certain boundary conditions. There are, however, two ways of attacking the problem: In the first method, one solves the field equations in general without referring to any excitation system, and then interprets the possible solutions (which are, in fact, infinite in number) according to the feasible physical arrangements. In the second method, one starts from the potential functions of some fundamental simple exciting systems (an electric or a magnetic dipole) and then generalizes them to fit the actual radiation systems. Mathematically, the first method consists in obtaining an immediate series expansion for each field intensity component with an undetermined (but unimportant) constant coefficient and finding the "characteristic values" of the argument of the cylindrical functions invovled according to the given boundary conditions. The properties of propagation can then be deduced from these "characteristic values" whose evaluation forms the kernel of interest. On the other hand, the second method requires some knowledge of the theory of functions of complex variables and the theory of integral equations in order to achieve the necessary transformations. Its apparently more difficult mathematical manipulations are, however, amply justified by the more comprehensive results obtained in a straightforward analytical manner. Nevertheless, the conclusions of both methods regarding the fundamental properties should be the same.

The first method (1) has been adopted by most physicists and engineers since the time of O. Heaviside and J. J. Thomson. It was, however, only in recent years that the problem of the hollow tube guide and the dielectric wire transmission has awakened some interest in the engineering field. The second method, which was first treated by R. Weyrich (2) in a formal mathematical manner, was, in fact, initiated by Sommerfeld (3) for the case of propagation of electromagnetic waves over the plane earth surface due to an electric dipole. The prime purpose of the present paper is to give a comprehensive physical discussion of the mathematical results obtained on the basis of the theory of functions. The detailed mathematical procedures, which can be found in the author's paper[4] presented to the California Institute of Technology, in part, also, in Weyrich's paper, are to be omitted here since they would obscure the conceptual gist of the subject.

It will be seen that, for distant transmission, the conclusion reached is rather unfavorable for a hollow tube guide and throws some light on the probable limitations of its applications.

In solving Maxwell's field equations for the electric and the magnetic field intensity components, we introduce suitable potential functions, which have only a z-component and whose wave equation in cylindrical coordinates becomes:

$$\nabla^2 u + k^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2} + k^2 u = 0 \tag{1}$$

The most general solution for the above wave equation has the following integral form (4), (5):

$$u = \int_L e^{\Omega s} e^{i\nu\varphi} [F_1(\lambda, \nu) R_\nu^{(1)}(r\sqrt{k^2 - \lambda^2}) + F_2(\lambda, \nu) R_\nu^{(2)}(r\sqrt{k^2 - \lambda^2})]\, d\lambda\, d\nu \tag{2}$$

where $F_1(\lambda, \nu)$ and $F_2(\lambda, \nu)$ are two arbitrary functions of λ and ν and L may be any suitably chosen four-dimensional region of the complex planes of λ and ν. The above expression (2) is the basic form of the second method capable of fitting not only the excitation system, but also the boundary conditions.

Part A

Characteristics of Propagation of a Linear Antenna

We shall discuss the case with the linear antenna situated along the axis of the hollow tube guide and consider the effect of its off-axis location upon the energy distribution at the end of this section. The potential function (or Hertzian function) for the space inside the tube guide for a linear antenna along the axis with center at origin can be shown to be:[1]

$$Re.[ue^{-i\omega t}] = Re.\left[\frac{i}{2}\int_{-\infty}^{\infty} e^{i(\lambda|z|-\omega t)} \cdot Q \cdot \frac{J_0(a\sqrt{k_1^2 - \lambda^2}) H_0^{(1)}(r\sqrt{k_1^2 - \lambda^2}) - H_0^{(1)}(a\sqrt{k_1^2 - \lambda^2}) J_0(r\sqrt{k_1^2 - \lambda^2})}{J_0(a\sqrt{k_1^2 - \lambda^2})}\, d\lambda\right]$$

$$= Re.\left[\frac{\pi}{a^2}\sum_{\nu=1}^{\infty} e^{i(\lambda_\nu|z|-\omega t)} \cdot Q \cdot \frac{x_\nu H_0^{(1)}(x_\nu)}{\lambda_\nu J_1(x_\nu)} J_0\left(x_\nu \frac{r}{a}\right)\right] \tag{3}$$

The last series expansion is obtained by means of the theory of residual calculus. It is rapidly convergent for all practical cases and is suitable for numerical computation. The function Q depends upon the harmonics of excitation. It is expressed by:

$$Q = \frac{2q\pi}{l} \frac{(-)^n \cos \lambda_\nu \frac{l}{2}}{\frac{q^2\pi^2}{l^2} - \lambda_\nu^2}, \qquad \left(|z| > \frac{l}{2}\right) \tag{4}$$

where $q = \begin{cases} 2n \ldots\ldots \text{for even harmonics,} \\ 2n+1 \ldots \text{for odd harmonics.} \end{cases}$

[1] For notations used, see Appendix I.

We have, in fact, a half-wavelength antenna for $q = 1$, a full-wavelength antenna for $q = 2$, etc.

From Equation (3), the magnetic and the electric field intensity components are obtained. They are:

$$\left.\begin{aligned} H_\varphi &= Re.\left[\frac{\pi}{\mu_1 a^3}\sum_{\nu=1}^{\infty} e^{i(\lambda_\nu|z|-\omega t)}\cdot Q\cdot\frac{x_\nu^2 H_0^{(1)}(x_\nu)}{\lambda_\nu J_1(x_\nu)}\,J_1\left(x_\nu\frac{r}{a}\right)\right] \\ H_z &= H_r = 0 \\ E_r &= Re.\left[\frac{\pi\omega}{ca^3 k_1^2}\sum_{\nu=1}^{\infty} e^{i(\lambda_\nu|z|-\omega t)}\cdot Q\cdot\frac{x_\nu^2 H_0^{(1)}(x_\nu)}{J_1(x_\nu)}\,J_1\left(x_\nu\frac{r}{a}\right)\right] \\ E_z &= Re.\left[\frac{i\pi\omega}{ca^4 k_1^2}\sum_{\nu=1}^{\infty} e^{i(\lambda_\nu|z|-\omega t)}\cdot Q\cdot\frac{x_\nu^3 H_0^{(1)}(x_\nu)}{\lambda_\nu J_1(x_\nu)}\,J_0\left(x_\nu\frac{r}{a}\right)\right] \\ E_\varphi &= 0 \end{aligned}\right\} \tag{5}$$

With these explicit expressions for all the field intensity components, we are in a position to discuss the properties of propagation in a quite rigorous and comprehensive way.

(1) "Distinct Modes" of Propagation

In Equations (3)–(5), the x_ν's are the distinct roots of

$$J_0(x) = J_0(a\sqrt{k_1^2 - \lambda^2}) = 0$$

and the λ_ν's are the corresponding propagation constants. Consequently, every such root gives rise to a "distinct mode" of propagation with its attenuation, phase relation and velocity different from all others. The resultant field at any point along the system is thus a superposition of all these "modes," while each "mode" propagates down the tube guide as if it existed alone. There would be no interaction between these different modes if the guide system were a perfect circular cylinder and had a homogeneous electric property throughout. With the above visualization, it suffices to discuss the characteristics of propagation of any of these modes.

(2) Attenuation Constant and Cut-off Frequency

The complex propagation constant λ_ν, for the νth mode, say, has a real and an imaginary component. It is expressed by:

$$\lambda_\nu = \beta_\nu + i\alpha_\nu \tag{6}$$

α_ν is then the attenuation constant and β_ν the phase constant by definition. Using Poynting's theorem for the energies propagated in the axial and in the radial direction, we obtain, after some mathematical transformations, the following explicit expression for the attenuation constant of the "νth mode":

$$\alpha_\nu = \frac{\mu_2 c k_1^2}{2\mu_1 a\beta_\nu\sqrt{2\pi\omega\mu_2\sigma_2}} = \frac{1}{2a}\sqrt{\frac{\mu_2\epsilon_1}{\mu_1\sigma_2}}\,\frac{f^{1/2}}{\sqrt{1-\left(\frac{f_\nu}{f}\right)^2}} \tag{7}$$

where

$$f_\nu = \frac{x_\nu c}{2\pi a \sqrt{\epsilon_1 \mu_1}} \tag{8}$$

is the cut-off frequency corresponding to the νth mode. We have, however, the νth mode propagation only when the applied frequency f is greater than f_ν. If the excitation frequency f is decreased below f_ν, all higher modes than the νth mode and the νth mode itself disappear. The total propagation consists then of a finite number of modes. For an uniform and homogeneous transmission system, these different modes propagate down the axis independently from each other. We must, therefore, calculate the attenuation constant for each mode separately and add up the effect due to all the modes at the point of reception by super-position.

(3) Phase Constant, Phase Velocity, and Group Velocity

Since the attenuation constant is very small except when the excitation frequency becomes extremely high, the phase constant is practically given by:

$$\beta_\nu = \sqrt{k_1^2 - \frac{x_\nu^2}{a^2}} = \frac{2\pi f \sqrt{\epsilon_1 \mu_1}}{c} \sqrt{1 - \left(\frac{f_\nu}{f}\right)^2} \tag{9}$$

The corresponding phase velocity for the νth mode is:

$$v_{p\nu} = \frac{\omega}{\beta_\nu} = \frac{c}{\sqrt{\epsilon_1 \mu_1} \sqrt{1 - \left(\frac{f_\nu}{f}\right)^2}} \tag{10}$$

It is therefore infinite when the applied frequency is equal to the cut-off value ($f = f_\nu$) and decreases monotonically as the frequency increases, approaching the velocity of light in the air medium as a limit at $f = \infty$. There are as many distinct phase velocities as there are "distinct modes" of propagation.

The group velocity,[2] which is of importance when modulated signals are to be transmitted, can be obtained from Equation (9) by differentiating ω against β_ν, keeping in mind that $\frac{x_\nu^2}{a^2}$ is a constant for the νth mode, we obtain:

$$v_{g\nu} = \frac{d\omega}{d\beta_\nu} = \frac{c}{\sqrt{\epsilon_1 \mu_1}} \sqrt{1 - \left(\frac{f_\nu}{f}\right)^2} \tag{11}$$

It begins from zero at the cut-off frequency (f_ν) and monotonically increases with the frequency, also approaching the velocity of light as a limit. From the above equation, we see that a simple relation exists between the phase velocity and the group velocity as follows:

$$v_{g\nu} v_{p\nu} = \frac{c^2}{\epsilon_1 \mu_1} \tag{12}$$

It is independent of the mode of propagation. There are also as many distinct group velocities as there are distinct modes of propagation.

[2] This can be referred to in any textbook on optics.

(4) Frequency Spectrum

From what has been discussed before, one notices that corresponding to each mode both the attenuation and the velocity of propagation are functions of frequency. But, however, even for a wide band modulated signal in television, which occupies about a six million cycle band, the variation of attenuation and velocity for each mode is practically negligible as can be seen from the curves plotted (Appendix II).

The frequency characteristics are distinct for the distinct modes of propagation if the transmission system is uniform and homogeneous. When the applied frequency f is higher than the first cut-off frequency $f_1 = \frac{2.4048c}{2\pi a\sqrt{\epsilon_1\mu_1}}$, but lower than the second cut-off frequency $f_2 = \frac{5.5201c}{2\pi a\sqrt{\epsilon_1\mu_1}}$, then only one single "mode" of propagation exists. For a television signal, if the modulated frequency band lies completely within f_1 and f_2, then there is only one single "mode" for the whole band. If the modulated signal lies between f_2 and $f_3 = \frac{8.6537c}{2\pi a\sqrt{\epsilon_1\mu_1}}$, there would be two distinct modes propagated with different attenuations and velocities. So with the whole modulated signal within f_n and f_{n+1}, there would be n distinct modes of propagation. Upon reception, these different modes are then superposed on each other, *but with their amplitude and phase relations widely different from those at the transmitter end.* When the transmission distance is long, *these different modes would give rise to a serious distortion of the original composition.* The possibility of using a suitable device to filter out all modes except the one desired might be conjectured at first sight; but unfortunately these different modes do not mean different bands of frequency in the usual sense. We have, in fact, still only one frequency band lying, say, between f_n and f_{n+1}, but the total energy emitted is divided among the n modes which propagate down the tube with different attenuations and velocities. Filter action can only be effected among different frequency bands from the exciting systems, not among those different modes ensuing from merely an energy distribution. Thereby a unique conclusion is reached: *That is, we must have "single mode" transmission.* In other words, we must design the excitation and the transmission system in such a way so that the whole modulated band lies between f_1 and f_2.

Considering the case of a cylindrical metal tube guide with air medium and an inner radius of 10 centimeters, we have then:

$f_1 = 1.148 \times 10^9$ cycles per sec.

$f_2 = 2.636 \times 10^9$ cycles per sec.

and the allowed frequency band is:

$\Delta f = f_2 - f_1 = 1.488 \times 10^9$ cycles per sec.,

which is amply sufficient for television and other purposes.

The above conclusion is not only of practical importance, but also of theoretical interest. Since then in expressions (3) and (5) we need only consider the first term of the infinite series.

(5) Characteristic Impedance and Radiation Resistance

The conception of characteristic impedance is helpful due to its frequent occurrence in the conventional electric circuit theory. It is always defined with respect to some current flowing along the transmission circuit. So we shall follow the same general procedure in defining the "characteristic impedance" (6) offered by an infinite cylindrical hollow tube guide to a linear antenna as the "ratio of the time averaged surface integration of the complex Poynting's vector to the time averaged square of the current flowing in the metal sheath in the axial or z-direction." The complex Poynting's vector is:

$$\mathbf{N} = \frac{c}{4\pi}[\tfrac{1}{2}\mathbf{E} \times \mathbf{H}] \tag{13}$$

Its surface integration over the enclosing surface of a cylinder of unit length fitting the inner surface of the hollow tube guide and the transformed volume integral over the volume of the same cylinder are given by:

$$\left.\begin{aligned} \bar{\mathbf{P}} &= \frac{-c}{4\pi}\int_s [\tfrac{1}{2}\mathbf{E} \times \mathbf{H}]\cdot d\mathbf{S} \\ &= \int_v \sigma \tfrac{1}{2}\mathbf{E}\cdot\mathbf{E}^* d\tau + 2i\omega \int_v \left[\frac{\mu}{8\pi}\frac{1}{2}\mathbf{H}\cdot\mathbf{H}^* - \frac{\epsilon}{8\pi}\frac{1}{2}\mathbf{E}\cdot\mathbf{E}^*\right] d\tau \\ &= \overline{W} + 2i\omega[\overline{U}_{\text{mag}} - \overline{U}_{\text{el}}] \end{aligned}\right\} \tag{14}$$

* represents the complex conjugate.

$\bar{P}$ is therefore a time averaged complex quantity whose real part represents the mean Joule heat developed per second and whose imaginary part is twice the amount of the difference of the mean magnetic energy and the mean electric energy multiplied by ω. Thus, if $\mathbf{I}$ is the total current of the system, and $Z_\nu = R_\nu + iX_\nu$ is the characteristic impedance, we have then:

$$\tfrac{1}{2}(\mathbf{I}\cdot\mathbf{I}^*)Z_\nu = \bar{P}_\nu \tag{15}$$

This is for the νth mode only. There are as many distinct characteristic impedances as there are distinct modes of propagation. The solenoidal current $\mathbf{I}$ is defined by the following relation:

$$\int_0^{2\pi} H_\varphi a\, d\varphi = \frac{4\pi \mathbf{I}}{c} \tag{16}$$

where H_φ is given in Equations (5). Carrying out the above indicated calculation, we obtain, for the νth mode, the following expressions for the real and the imaginary component of Z_ν. They are:

$$\left.\begin{aligned} R_\nu &= \frac{2\beta_\nu}{\omega\epsilon_1} = \frac{2}{c}\sqrt{\frac{\mu_1}{\epsilon_1}}\sqrt{1 - \left(\frac{f_\nu}{f}\right)^2} \\ X_\nu &= 0 \end{aligned}\right\} \tag{17}$$

The characteristic impedance has practically only a pure resistive component, which might be called the radiation resistance of the system. There are as many distinct radiation resistances as there are distinct modes of propagation. Considering the first mode only, the radiation resistance R_1 is zero at the cut-off frequency, increases rapidly to $R_1 = 490$ ohms at point of minimum attenuation and reaches the value $R = 600$ ohms at $f = f_2$. The variation of the characteristic impedance is therefore very small for the whole usable range of frequencies. For a band of a few megacycles per second, it is practically constant.

(6) Field Structure

We shall limit the study to the field structure in the dielectric (air) medium inside the hollow tube guide, which is electrically thick and is only penetrated to a very small distance from the inner surface by the high frequencies involved. From Equations (5) we see that all the field intensity components have a slightly damped sinusoidal distribution along the axial- or z-direction. The spatial wavelength in the air medium is given by:

$$L_\nu = \frac{2\pi}{\beta_\nu} = \frac{c}{\sqrt{\epsilon_1 \mu_1} f \sqrt{1 - \left(\frac{f_\nu}{f}\right)^2}} = \frac{L_0}{\sqrt{1 - \left(\frac{f_\nu}{f}\right)^2}} \tag{18}$$

where L_0 is the free space wavelength for the case when the tube guide is absent. This apparent wavelength for any mode is therefore a function of the frequency and there are as many distinct wavelengths as there are modes of propagation.

The equations for the lines of magnetic intensity (H_φ) are concentric circles around the axis and the distribution in the radial- or r-direction is proportional to $J_1\left(x_\nu \frac{r}{a}\right)$. Consequently, for the range $0 \leqq r \leqq a$, there are ν-fixed nodes: We have then a standing wave in the dielectric (air) space with an amplitude varying sinusoidally with time.

The equations for the lines of electric intensity can be obtained by solving the differential equation:

$$\frac{dz}{E_z} = \frac{dr}{E_r}$$

Substituting E_z and E_r from Equations (11) into the above expression, and solving, we have:

$$\left[\frac{x_\nu r}{a} J_1\left(x_\nu \frac{r}{a}\right)\right][\cos(\beta_\nu |z| - \omega t)] = C \tag{19}$$

where the integration constant C must be determined from the given exciting system. The first term $\left[\frac{x_\nu r}{a} J_1\left(x_\nu \frac{r}{a}\right)\right]$ represents a standing wave in the radial- or r-direction, and the second term $\cos(\beta_\nu |z| - \omega t)$ indicates that the wave is propagated down the tube and at any fixed time the distribution is sinusoidal.

Since the field configuration is independent of the azimuthal angle φ, Equation (19) holds for all planes passing through the axis.

Any attempt to plot the electric lines must be based upon Equation (19).

(7) Terminal Devices

The preceding discussion has been based upon the case of an infinitely long metal tube. In any practical set-up, however, the transmitting and the receiving ends must be terminated by some devices, which, of course, should be so designed as to increase the efficiency of transmission and reception.

For a linear axial antenna, the simplest efficient termination is a closed end made of perfectly conducting material. The existence of such an end simply introduces an image situated symmetrically at the same distance behind the terminating surface as the exciter is situated in front of it. Then the resultant potential function (Eq. (3)), for a quarter-wavelength antenna with base on the terminating end, becomes that corresponding to a half-wavelength.

$$Re\cdot[ue^{-i\omega t}] = Re\cdot\left[\frac{\pi}{a^2}\sum_{\nu=1}^{\infty} e^{i(\Omega_\nu|z|-\omega t)}\cdot\frac{2\pi\cos\left(\lambda_\nu\frac{l}{2}\right)}{l\left(\frac{\pi^2}{l^2}-\lambda_\nu^2\right)}\cdot\frac{x_\nu H_0^{(1)}(x_\nu)}{\lambda_\nu J_1(x_\nu)}J_0\left(x_\nu\frac{r}{a}\right)\right] \quad (20)$$

For a half-wavelength antenna situated close to the end, we simply replace the factor $e^{\Omega_\nu z}$ in Equation (3) by $e^{\Omega_\nu z}\cdot(1+e^{\Omega_\nu l})$ and modify the factor Q properly. Such a type of ending is considered as the simplest and at the same time the most efficient. Other irregular terminal devices disturb the field configuration and make any attempt for rigorous analysis impossible.

The terminal device to be used at the receiving end is a much more difficult problem. Since the energy is propagated down the tube guide in the air medium, the characteristic impedance defined above has not the same sense as that in the conventional electric circuit theory, for which the energy (at least the major part) is carried by the movement of the electrons along the metallic conductors. For a hollow tube guide, it is almost impossible, at least at the present stage of knowledge, to design any mechanical or electrical device to absorb completely the energy in the air medium. One can understand this better by conjecturing that a hollow tube guide transmission is strictly a radio-communication with all energy enclosed within a cylindrical boundary, while a concentric cable (or any conventional circuit) transmission is basically a wire-transmission. Their essential difference is independent of the frequency used. If one considers how small a fraction of the energy in free space can be picked up by an ordinary radio receiving antenna, one realizes that no device, whatsoever, can absorb all the energy propagated in a hollow tube guide. An incomplete adsorption means reflection and causes interferences between the transmitter and the receiver. This forms, therefore, one very important factor which places a hollow guide system in a definitely inferior position as compared to a concentric cable system.

(8) Stability Problem

The stability problem arises when the transmission system becomes heterogeneous or deviating from the ideal straight, circular cylindrical tube. This apparently peculiar phenomenon is simply due to the fact that the metal tube guide itself really forms a continuous secondary radiation system by induction from the primary source. For long distance transmission, the two unavoidable deviations from the ideal system are: (a) Cross-section not being circular all along the length, some portion may assume oval or elliptic shape; and (b) bending of the tube at certain locations as found necessary in installation. No rigorous mathematical formulation is possible for any such transition. The following paragraph gives, therefore, a qualitative description of the physical phenomena.

For a linear antenna, slight deviation from circular cross-section is of no serious consequence: although a small φ-component of electric intensity (E_φ) may be introduced at such locations, but the magnetic field remains essentially circular (H_φ); the attenuation constant and velocity of propagation might have undergone some slight modifications. A bending of the tube, however, at any angle, may introduce a strong new field configuration depending on a $e^{i\varphi}$-factor and many less intense field configurations depending on factors $e^{in\varphi}$ ($n = 2, 3, 4, \cdots$). (These kinds of field configurations with factors $e^{in\varphi}$ for $n = 1, 2, 3, \cdots$, are called by some writers E_1, E_2, E_3, $\cdots$ waves.) Every such factor brings about a series of numerous "modes" of propagation like that given in Equations (3) and (5). Since the energy emitted from the primary source is constant, any new field configuration causes a great reduction of the original symmetrical field intensity. It is expected that the smaller the ratio of the radius of curvature and the wavelength, the more serious is the effect of bending.

The above qualitative analysis gives us a unique conclusion: that is, a hollow tube guide system is fundamentally an unstable system so far as the field configuration and energy distribution are concerned. This important consideration also places a hollow guide, for long distance transmission, in a very unfavorable position with respect to a concentric cable system. For the latter, no such question arises.

(9) Energy Distribution for Slight Off-axis Location of the Linear Exciter

It can be shown that, for a linear antenna of length l with center at $(0, r_0, \varphi_0)$, the expression for the potential function is:

$$Re.[ue^{-i\omega t}] = Re.\left[\frac{-1}{2}\sum_{m=-\infty}^{\infty} e^{im(\varphi-\varphi_0)}\sum_{s=0}^{\infty} e^{i(\lambda_{ms}z-\omega t)}\cdot Q \cdot \frac{x_{ms}J_m\left(x_{ms}\frac{r_0}{a}\right)Y_m(x_{ms})}{a^2\lambda_{ms}J_m'(x_{ms})}J_m\left(x_{ms}\frac{r}{a}\right)\right] \tag{21}$$

For a slight off-center location, that is, for $r_0 \ll a$ and $\frac{r_0}{a}x_{ms} \ll 1$ for the first few roots of $J_m(x_{ms}) = 0$, which and which only need be considered, the above expression reduces to:

$$Re.\,[ue^{-i\omega t}] = Re.\left[\frac{-1}{2}\sum_{m=-\infty}^{\infty} e^{im(\varphi-\varphi_0)}\sum_{s=0}^{\infty} e^{i(\lambda_{ms}z-\omega t)} \cdot \frac{x_{ms}^{m+1}\cdot QY_m(x_{ms})}{2^m m!\,a^2 J_m'(x_{ms})} J_m\left(\frac{r}{a}x_{ms}\right)\left(\frac{r_0}{a}\right)^m\right] \tag{22}$$

which is thus proportional to $\left(\frac{r_0}{a}\right)^m$ for all modes of propagation corresponding to $J_m(x) = 0$. The corresponding energy for each mode is, therefore, proportional to $\left(\frac{r_0}{a}\right)^m$. From this we get a fair picture of the energy distribution among the different modes. Or we may group together the energies for all modes for the same m under a single unit, then the energy unit distribution for different m's possesses approximately the following shape:

Curves (A) and (B) give a general idea of a "broad" and a "sharp" distribution, respectively. The smaller is the value $\left(\frac{r_0}{a}\right)$, the sharper is the distribution and the greater is the concentration of energy at $m = 0$. Curve (B) degenerates into curve (C) for the symmetrical case with the linear exciter located along the axis when the total energy emitted resides in the single unit $m = 0$.

Because in practical application we cannot use all the different modes of propagation with different attenuations and velocities (which, in fact, vitiates the reception as noted before), it is then evident that the axially symmetrical operation is the most efficient one for transmission and reception. This might be a conclusion of considerable importance for engineers who could finally bring the hollow guide system to a successful application in the future.

(10) Resonance, Maximum Response

One notices that, in the expressions (3) and (5) for the potential function and the field intensity components, there occurs a factor Q for a linear exciter system. It is capable of being adjusted to give maximum response.

The denominator of Q:

$$\frac{q^2\pi^2}{l^2} - \lambda_\nu^2 = \frac{q^2\pi^2}{l^2} - \left(k_1^2 - \frac{x_\nu^2}{a^2}\right) = \left(\frac{q^2\pi^2}{l^2} + \frac{x_\nu^2}{a^2} - \frac{\omega^2\epsilon_1\mu_1}{c^2}\right) - i\frac{4\pi\omega\sigma_1\mu_1}{c^2} \tag{23}$$

$$\approx \frac{x_\nu^2}{a^2}$$

is seen to be practically a real quantity, which can only be made small by increasing the radius of the hollow tube.

The numerator of Q, for a fixed frequency of excitation, follows a cosine curve with an increasing amplitude directly proportional to the length of the antenna.

This phenomena offers a very interesting interpretation. The amplitude becomes infinite when the antenna is infinitely long. This is a logical result, since an infinite antenna requires an infinite amount of energy of excitation. If, however, an outer return path is provided, the energy involved becomes finite and a hollow guide radiation transites into a two wire (concentric, say) transmission. The latter, in this sense, represents an optimum condition for a hollow guide: this conclusion, though unique in character, is of no engineering bearing, since we are here merely interested in getting maximum field intensity in a hollow guide radiation. We shall, therefore, assume q to be a finite quantity and find the condition for maximum $\cos\left(\frac{\lambda_\nu}{2} l\right)$. That is:

$$\lambda_\nu \frac{l}{2} = \sqrt{k_1^2 - \frac{x_\nu^2}{a^2}}\,\frac{l}{2} = m\pi \qquad (m = 1, 2, \cdots)$$

Neglecting the imaginary component of k_1^2, the above relation becomes:

$$\frac{4\pi^2 f^2 \epsilon_1 \mu_1}{c^2} - \frac{x_\nu^2}{a^2} = \frac{4\pi^2 m^2}{l^2} = \frac{4\pi^2 m^2 4f^2 \epsilon_1 \mu_1}{c^2 q^2}$$

or

$$\frac{4\pi^2 f^2 \epsilon_1 \mu_1}{c^2}\left(1 - \frac{4m^2}{q^2}\right) = \frac{x_\nu^2}{a^2} \qquad (24)$$

since $l = \frac{q}{2f\sqrt{\epsilon_1 \mu_1}}$. The possible realization of the above equation can best be illustrated by a numerical example.

For single mode propagation, with a carrier frequency of $f = 1.64 \times 10^9$ cycles per second and an inner radius of the hollow tube $a = 10$ cms., (24) specifies:

$$q^2 = 8.0m^2$$

For $m = 1$, this is approximated by $q = 3$, corresponding to odd harmonic excitation with the length of the antenna equal to 3 half-wavelengths. Thus we see that relation (24) can be exactly satisfied for $q = 3$, $a = 10$ cms., $m = 1$, by choosing a frequency $f \cong 1.546 \times 10^9$ cycles per second.

Part B

Characteristics of Propagation of a Loop Antenna

In this section, we shall limit the study to a loop antenna with a uniform current of excitation. The rigorous mathematical analysis is made possible by using the idea that a uniform current loop is equivalent to a plane magnetic shell with the same boundary. The expression for the "general electric vector potential"[3] for a plane magnetic shell is obtained by a surface integration of that of a magnetic dipole (7). It can be shown that for a circular uniform

[3] For this proposed name, see Appendix for its definition.

current loop of radius b with center on the axis of a hollow tube guide, the potential function for the air medium is given by:

$$Re.[ue^{-i\omega t}]$$

$$= \begin{cases} Re.\left[i\pi b \int_{-\infty}^{\infty} e^{i(\lambda z-\omega t)} \cdot \dfrac{J_1(a\sqrt{k_1^2-\lambda^2})H_1^{(1)}(b\sqrt{k_1^2-\lambda^2}) - J_1(b\sqrt{k_1^2-\lambda^2})H_1^{(1)}(a\sqrt{k_1^2-\lambda^2})}{\sqrt{k_1^2-\lambda^2}\,J_1(a\sqrt{k_1^2-\lambda^2})} \cdot J_0(r\sqrt{k_1^2-\lambda^2})\,d\lambda \right] & \text{for } r < b \\ Re.\left[i\pi b \int_{-\infty}^{\infty} e^{i(\lambda z-\omega t)} \cdot \dfrac{J_1(a\sqrt{k_1^2-\lambda^2})H_0^{(1)}(r\sqrt{k_1^2-\lambda^2}) - J_0(r\sqrt{k_1^2-\lambda^2})H_1^{(1)}(a\sqrt{k_1^2-\lambda^2})}{\sqrt{k_1^2-\lambda^2}\,J_1(a\sqrt{k_1^2-\lambda^2})} \cdot J_1(b\sqrt{k_1^2-\lambda^2})\,d\lambda \right] & \text{for } r > b \end{cases}$$

$$= Re.\left[\frac{-2\pi^2 b}{a} \sum_{\nu=1}^{\infty} e^{i(\lambda_\nu z-\omega t)} \cdot \frac{H_1^{(1)}(y_\nu)J_1\left(\frac{b}{a}y_\nu\right)}{\lambda_\nu \frac{d}{dy_\nu}[J_1(y_\nu)]} J_0\left(\frac{r}{a}y_\nu\right) \right] \tag{25}$$

The last expression is obtained by means of the theory of residual calculus from the integral equation. It might be noticed that the resultant series expansion has another additional term which is different for $r < b$ and $r > b$. Since it plays no role in propagation, it can be advantageously dispensed with. The y_ν's are the roots of $J_1(y) = 0$.

The expressions for the electric and magnetic field intensity components for the present symmetrical case become:

$$\left.\begin{aligned} E_\varphi &= Re.\left[\frac{-1}{\epsilon} \sum_{\nu=1}^{\infty} \frac{2\pi^2 b}{a} \left(\frac{y_\nu}{a}\right) \frac{H_1^{(1)}(y_\nu)J_1\left(\frac{b}{a}y_\nu\right)}{\lambda_\nu \frac{d}{dy_\nu}[J_1(y_\nu)]} J_1\left(\frac{r}{a}y_\nu\right) \cdot e^{i(\lambda_\nu z-\omega t)} \right] \\ E_z &= E_r = 0 \\ H_z &= Re.\left[\frac{-c}{i\omega\epsilon_1\mu_1} \sum_{\nu=1}^{\infty} \frac{2\pi^2 b}{a} (k_1^2-\lambda_\nu^2) \frac{H_1^{(1)}(y_\nu)J_1\left(\frac{b}{a}y_\nu\right)}{\lambda_\nu \frac{d}{dy_\nu}[J_1(y_\nu)]} J_0\left(\frac{r}{a}y_\nu\right) e^{i(\lambda_\nu z-\omega t)} \right] \\ H_r &= Re.\left[\frac{c}{\omega\epsilon_1\mu_1} \sum_{\nu=1}^{\infty} \frac{2\pi^2 b}{a} \left(\frac{y_\nu}{a}\right) \frac{H_1^{(1)}(y_\nu)J_1\left(\frac{b}{a}y_\nu\right)}{\frac{d}{dy_\nu}[J_1(y_\nu)]} J_1\left(\frac{r}{a}y_\nu\right) e^{i(\lambda_\nu z-\omega t)} \right] \\ H_\varphi &= 0 \end{aligned}\right\} \tag{26}$$

The following description of the properties of propagation for a circular current loop follows the same line as that discussed in the preceding section for a linear antenna. Many points of interest and their procedures of derivation are identical for the two cases. For these we shall omit the detailed description here and suggest to the reader to read this section in parallel with the previous section A.

(1) "Distinct Modes" of Propagation

Referring to Equations (25) and (26), one notices that the "distinct modes" of propagation for the present case correspond to the distinct roots of $J_1(y) \equiv J_1(a\sqrt{k_1^2 - \lambda^2}) = 0$. These modes have different attenuations and velocities and would propagate down the tube guide independently of each other if the transmission system is perfectly uniform and homogeneous. Therefore it suffices to discuss the characteristics of propagation of any of these modes.

(2) Attenuation Constant and Cut-off Frequency

The complex propagation constant $\lambda_\nu = \beta_\nu + i\alpha_\nu$ is given by the relation $\lambda_\nu^2 = \left(k_1^2 - \frac{y_\nu^2}{a^2}\right)$. The attenuation constant α_ν is, however, calculated by Poynting's theorem as before and its explicit expression is found to be:

$$\alpha_\nu = \frac{\mu_2 c y_\nu^2}{4\mu_1 a^3 \beta_\nu \sqrt{2\pi\omega\mu_2\sigma_2}} = \frac{1}{4a}\sqrt{\frac{\mu_2\epsilon_1}{\mu_1\sigma_2}}\,\frac{f_\nu^2}{f^{3/2}\sqrt{1 - \left(\frac{f_\nu}{f}\right)^2}} \tag{27}$$

where $f_\nu = \dfrac{y_\nu c}{2\pi a\sqrt{\epsilon_1\mu_1}}$

is the cut-off frequency corresponding to the νth mode. Both the attenuation constant and the cut-off frequency are distinct for the "distinct modes" of propagation. Comparing Equation (27) with Equation (7), one notices that the attenuation constant for a current loop follows a different law from that for a linear antenna. In fact, it decreases monotonically from the value of infinity at $f = f_\nu$ to the value zero at $f = \infty$. This peculiar character, however, cannot be utilized to full advantage from both the theoretical and the engineering points of view. First, the whole modulated frequency band must be confined within

$$f_1 = \frac{y_1 c}{2\pi a\sqrt{\epsilon_1\mu_1}} = \frac{3.8317c}{2\pi a\sqrt{\epsilon_1\mu_1}} \quad \text{and} \quad f_2 = \frac{y_2 c}{2\pi a\sqrt{\epsilon_1\mu_1}} = \frac{7.0156c}{2\pi a\sqrt{\epsilon_1\mu_1}}$$

for single mode of transmission; second, the generation of extremely high frequencies is ever increasingly difficult. The slightly lower attenuation of (27) compared with (7) is still an advantage for any future possibility of long distance transmission.

(3) Phase Constant, Phase Velocity, and Group Velocity

The discussion given in (A) (3), applies here word for word with the only modification that the cut-off frequency for the νth mode is calculated from:

$$f_\nu = \frac{y_\nu c}{2\pi a\sqrt{\epsilon_1\mu_1}}$$

(4) Frequency Spectrum

Identical reasoning follows from (A) (4). The same conclusion about the single mode transmission is reached.

Considering the case of a cylindrical metal tube guide with air medium and an inner radius of 10 centimeters, the frequency limits for the single mode transmission are:

$$f_1 = \frac{3.8317 \times 3 \times 10^{10}}{2\pi 10} = 1.83 \times 10^9 \text{ cycles per second}$$

$$f_2 = \frac{7.0156 \times 3 \times 10^{10}}{2\pi 10} = 3.35 \times 10^9 \text{ cycles per second}$$

The allowed frequency band is then:

$$\Delta f = f_2 - f_1 = 1.52 \times 10^9 \text{ cycles per second,}$$

which is about two percent wider than that for a linear antenna.

In Equations (25) and (26), we need also only consider the first term of the the infinite series.

(5) Characteristic Impedance and Radiation Resistance

These will not be discussed, since for a loop antenna, the definition is artificial and its physical significance not evident.

(6) Field Structure

The discussion in (A) (6) holds here also word for word, with the following modifications due to the interchange of the roles of the electric and the magnetic field intensity components.

Referring to Equations (26), we see that for a current loop the lines of electric intensity (E_φ) form concentric circles around the axis and the distribution in the radial- or r-direction is proportional to $J_1\left(\frac{r}{a} y_\nu\right)$. For the range $0 \leqq r \leqq a$, we have here $(\nu + 1)$ nodes, since y_ν's are the roots of $J_1(y) = 0$. Therefore, a standing wave is formed in the radial direction with its amplitude varying sinusoidally with time, but with its $(\nu + 1)$ nodes fixed in position.

The differential equation for the lines of magnetic field intensity is:

$$\frac{dz}{H_z} = \frac{dr}{H_r}$$

Substituting H_z and H_r from Equations (26), and solving, we obtain a similar result:

$$\left[y_\nu \frac{r}{a} \cdot J_1\left(y_\nu \frac{r}{a}\right)\right][\cos(\lambda_\nu |z| - \omega t)] = c, \tag{28}$$

where the constant of integration, C, has to be determined from the excitation strength. Plotting of the magnetic lines should be based upon Equation (28).

(7) Terminal Devices

Referring to Equation (25), the resultant potential function for a uniform current loop situated at a distance ζ from the conducting closed end becomes:

$$\text{Re.}[ue^{-i\omega t}] = \text{Re.}\left[-2i\sum_{\nu=1}^{\infty}\frac{2\pi^2 b}{a}\frac{H_1^{(1)}(y_\nu)J_1\left(y_\nu\frac{b}{a}\right)}{\lambda_\nu\frac{d}{dy_\nu}[J_1(y_\nu)]}J_0\left(y_\nu\frac{r}{a}\right)\sin\lambda_\nu\zeta\cdot e^{i(\lambda_\nu z-\omega t)}\right] \quad (29)$$

For maximum reinforcing effect, we should adjust the distance ζ so that:

$$\lambda_\nu\zeta = \sqrt{k_1^2-\frac{y_\nu^2}{a^2}}\cdot\zeta = \sqrt{\frac{\omega^2\epsilon_1\mu_1}{c^2}-\frac{y_\nu^2}{a^2}} = \frac{\pi}{2}(2n+1)$$

holds, then one has:

$$\sin\lambda_\nu\zeta = (-)^n.$$

The discussion given in (A) (7), applies here word for word and will not be repeated.

(8) Stability Problem

Remarks in (A) (8) can be used here with the interchange of the roles of the electric and the magnetic field components. The other modifications are: additional new field configurations produced by any deviation of the cross-section from a circular one and by bending of the system for long distance transmission introduce factors $e^{in\varphi}$ ($n = 1, 2, 3, \cdots$), which some writers call the H_1, H_2, H_3, $\cdots$, waves.

(9) Energy Distribution for Slight Off-axis Location of the Loop Antenna

Similar description and conclusion follow from (A) (9).

(10) Maximum Response

For a loop antenna with uniform current distribution, there is no such resonance as that discussed in (A) (10) for a harmonically excited linear antenna.

Referring to Equation (25), one observes that the only possible adjustment of the dimension of the loop antenna relates itself to the factor $bJ_1\left(y_\nu\frac{b}{a}\right)$, which should be made a maximum. That is; b should satisfy the following relation:

$$\frac{d}{db}\left[bJ_1\left(y_\nu\frac{b}{a}\right)\right] = J_1\left(y_\nu\frac{b}{a}\right)+y_\nu\frac{b}{a}J_1'\left(y_\nu\frac{b}{a}\right) = 0$$

or

$$J_0\left(y_\nu\frac{b}{a}\right) = 0 \quad (30)$$

Take the case of single mode transmission, we must have $\left(y_1 \frac{b}{a}\right)$, a root of $J_0(x) = 0$. That is:

$$3.8317 \frac{b}{a} = x_1 = 2.4048$$

or

$$b = 0.629a \tag{31}$$

The above relation might be of use to the designing engineers.

Concluding Remarks

From the above discussions one notices that for long distance transmission a hollow tube guide is a fundamentally unstable system. The variations of the field configurations with the corresponding changes in velocities and attenuations are almost unavoidable from practical considerations. Further, no efficient receiving terminating device can be obtained and reflection and interference ensue from any kind of termination. All these fundamental considerations tend to put the hollow guide system in a very unfavorable position in technical applications in spite of certain fascinating aspects of its peculiar properties.

A comparison of the properties of propagation between a hollow tube guide and a concentric system will be treated in another paper.

APPENDIX I

Notations

Gaussian units are used throughout in this treatment.

$\mathbf{H}$ = Vector magnetic field intensity with components H_z, H_r, and H_φ.
$\mathbf{E}$ = Vector electric field intensity with components E_z, E_r, and E_φ.
ϵ = Dielectric constant.
μ = Permeability.
σ = Conductivity.
c = Velocity of light.
$\mathbf{U}$ = General magnetic vector potential with components u_z, u_r, and u_φ.
Relation of definition: $\mu\mathbf{H} = \nabla \times \mathbf{U}$
$\mathbf{U}_m$ = General electric vector potential with components u_{mz}, u_{mr}, and $u_{m\varphi}$
Relation of definition: $\epsilon\mathbf{E} = \nabla \times \mathbf{U}_m$
z, r, φ = Cylindrical coordinates of field point.
ζ, r_0, φ_0 = Cylindrical coordinates of source point.
Maxwell's field equations in vector form:

$$\nabla \times \mathbf{H} = \frac{1}{c}\left(4\pi\sigma\mathbf{E} + \epsilon \frac{\partial \mathbf{E}}{\partial t}\right)$$

$$\nabla \times \mathbf{E} = -\frac{\mu}{c} \frac{\partial \mathbf{H}}{\partial t}$$

$$\nabla \cdot (\mu\mathbf{H}) = 0, \qquad \nabla \cdot (\epsilon\mathbf{E}) = 4\pi\rho$$

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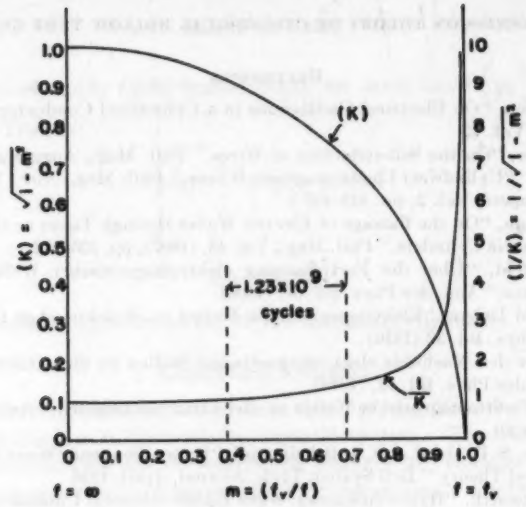
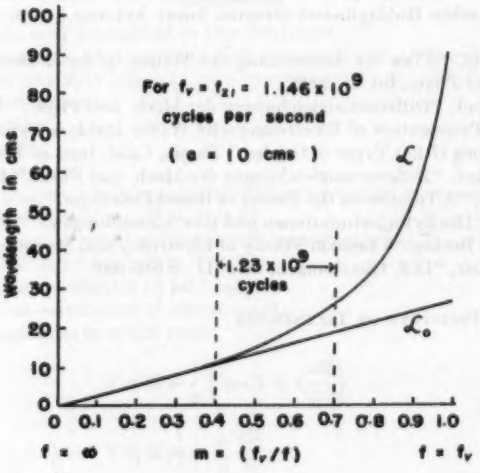


GRAPH 1

Phase Velocity $$V_{pv} = \frac{c}{\sqrt{\epsilon_1 \mu_1}} \frac{1}{\sqrt{1 - (f_v/f)^2}} = \frac{c}{\sqrt{\epsilon_1 \mu_1}} \frac{1}{K} \tag{10}$$

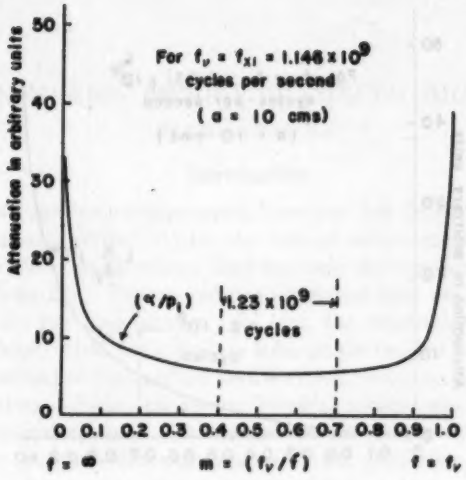
Group Velocity $$V_{gv} = \frac{c}{\sqrt{\epsilon_1 \mu_1}} \sqrt{1 - (f_v/f)^2} = \frac{c}{\sqrt{\epsilon_1 \mu_1}} K \tag{11}$$

Radiation Resistance $$R_v = \frac{2}{c} \sqrt{\frac{\mu_1}{\epsilon_1}} \sqrt{1 - (f_v/f)^2} = \frac{2}{c} \sqrt{\frac{\mu_1}{\epsilon_1}} K \tag{17}$$



GRAPH 2

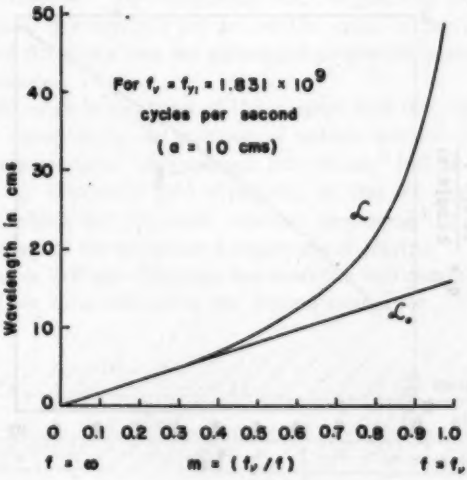
Wavelength $$\mathfrak{L}_v = \mathfrak{L} = \frac{c}{\sqrt{\epsilon_1 \mu_1}} \frac{1}{f} \frac{1}{\sqrt{1 - (f_v/f)^2}} = \frac{c}{\sqrt{\epsilon_1 \mu_1}} \frac{1}{f} \frac{1}{K} = \mathfrak{L}_0 \frac{1}{K} \tag{18}$$


Attenuation in arbitrary units
For $f_v = f_{x1} = 1.148 \times 10^9$ cycles per second
(a = 10 cms)
(α/p_1)
1.23×10^9 cycles
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0
f = ∞
m = (f_v/f)
f = f_v

GRAPH 3

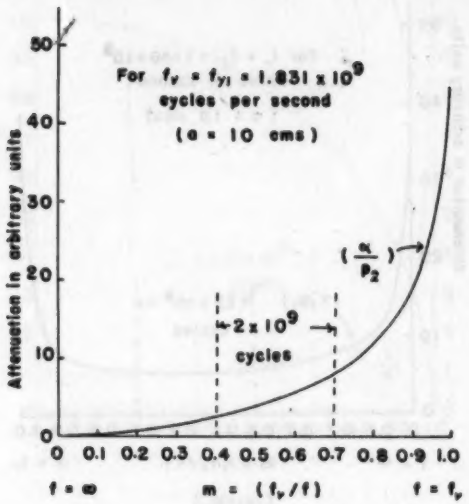
$$\text{Attenuation } \alpha_v = \frac{1}{2a}\sqrt{\frac{\mu_2 \epsilon_1}{\mu_1 \sigma_2}}\,\frac{f^4}{\sqrt{1-(f_v/f)^2}} = p_1\,\frac{f^4}{10^4 K} \qquad (7)$$

$$p_1 = \frac{10^4}{2a}\sqrt{\frac{\mu_2 \epsilon_1}{\mu_1 \sigma_2}}$$


Wavelength in cms
For $f_v = f_{y1} = 1.831 \times 10^9$ cycles per second
(a = 10 cms)
$\mathfrak{L}$
$\mathfrak{L}_0$
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0
f = ∞
m = (f_v/f)
f = f_v

GRAPH 4

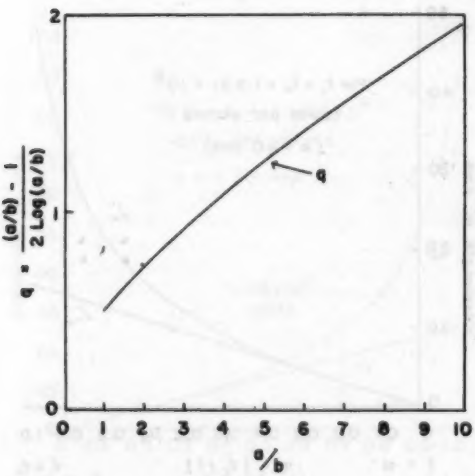
$$\text{Wavelength } \mathfrak{L}_v = \mathfrak{L} = \frac{c}{\sqrt{\epsilon_1 \mu_1}}\,\frac{1}{f}\,\frac{1}{\sqrt{1-(f_v/f)^2}} = \frac{c}{\sqrt{\epsilon_1 \mu_1}}\,\frac{1}{f}\,\frac{1}{K} = \mathfrak{L}_0\,\frac{1}{K}$$


For $f_v = f_{y1} = 1.831 \times 10^9$
cycles per second
(a = 10 cms)
Attenuation in arbitrary units
$(\frac{\alpha}{p_2})$
2×10^9
cycles
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0
f = ∞
m = (f_v/f)
f = f_v

GRAPH 5

$$\text{Attenuation } \alpha_v = \frac{1}{4a}\sqrt{\frac{\mu_2 \epsilon_1}{\mu_1 \sigma_2}}\,\frac{f^3(f_v/f)^2}{\sqrt{1-(f_v/f)^2}} = p_2\frac{10f^3(f_v/f)^2}{K} \qquad (27)$$

$$p_2 = \frac{1}{40a}\sqrt{\frac{\mu_2 \epsilon_1}{\mu_1 \sigma_2}}$$



GRAPH 6

$$q = \lim_{f>>f_v}\frac{\alpha}{\alpha_v} = \frac{(a/b)-1}{2\,\mathrm{Log}\,(a/b)} \qquad (25)$$

TRANSMISSION THEORY OF CONCENTRIC LINES

By Chang-Pen Hsü

Introduction

The theory of concentric transmission lines was first treated comprehensively by J. R. Carson and Gilbert (1) for the case of submarine cables. The same general method seems to have been used by many authors for different types of concentric systems (2). The procedures employed here are, however, a little different from the previous articles. In fact, the development of the present essay follows closely that for a hollow tube guide treated by the author (3), following the method of Sommerfeld and Weyrich, from the point of view of the theory of functions. Some interesting physical aspects are brought out in a formal mathematical manner. The simple results obtained here are specially suitable for the purpose of comparison with the properties of a hollow tube guide.

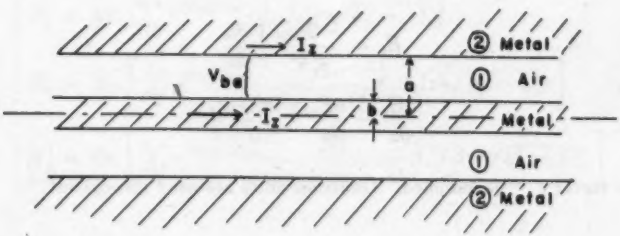
Mathematical Theory

The prime purpose of this section is to present some simple and rigorous expressions for the propagation constant of a concentric transmission system so that a clear view of the relative merits of the concentric system compared with a hollow tube guide can be grasped. By means of the asymptotic properties of the cylindrical functions, the desired results are obtained in a simple and straightforward manner. By comparing with the previous works on the theory of concentric lines, one can not fail to see the value of the intrinsic character of the cylindrical functions and the correlated properties based upon the theory of complex functions.

Because of the explicit relations of the current and the voltage between the two concentric conductors, the method of attack follows in a general sense that used for "conventional transmission line circuit" but at the same time we shall be guided by Maxwell's field equations, so that we shall be aware of the approximations which can be made without impairing the accuracy desired. The circuit diagram is shown below for purpose of clarity.

V_{ba} = voltage difference between the two conductors

I_z = total current in the central conductor



The current in the central conductor of radius b is assumed to be in the positive z-direction and the return path is formed by the outside hollow tube with an inner radius a. For the present symmetrical field configuration, Maxwell's equations reduce to*

$$(1)\quad \begin{cases} \dfrac{1}{r}\dfrac{\partial(rH_\varphi)}{\partial r} = \dfrac{4\pi\sigma}{c}E_z + \dfrac{\epsilon}{c}\dfrac{\partial E_z}{\partial t} = \dfrac{ck^2}{i\omega\mu}E_z \\ -\dfrac{\partial H_\varphi}{\partial z} = \dfrac{4\pi\sigma}{c}E_r + \dfrac{\epsilon}{c}\dfrac{\partial E_r}{\partial t} = \dfrac{ck^2}{i\omega\mu}E_r \\ \dfrac{\partial E_r}{\partial z} - \dfrac{\partial E_z}{\partial r} = -\dfrac{\mu}{c}\dfrac{\partial H_\varphi}{\partial t} = \dfrac{i\omega\mu}{c}H_\varphi \end{cases}$$

where $\mathbf{E} = E_r, E_z, E_\varphi = 0; \mathbf{H} = H_\varphi, H_z = H_r = 0$. The assumptions to be made in the present analysis can be summarized as follows:

(a) $k_1^2 = \dfrac{\omega^2\epsilon_1\mu_1 + i4\pi\omega\mu_1\sigma_1}{c^2} \approx \dfrac{\omega^2\epsilon_1\mu_1}{c^2}, \qquad \sigma_1 \to 0$

(b) $k_2^2 = \dfrac{\omega^2\epsilon_2\mu_2 + i4\pi\omega_2\sigma_2}{c^2} \approx \dfrac{i4\pi\omega\mu_2\sigma_2}{c^2}, \qquad \sigma_2 \to \text{large}$

(c) outside conductor electrically thick.

These coincide with that used for the discussion of a hollow guide system and are nearly realized in practice. The case that σ_1 is not equal to zero can be taken into account very easily.

Now we shall find the relations between the current I_z, the voltage V_{ba} and the field intensities. i_z is used to represent the current enclosed within a circular crosssection of radius r and I_z that in the central conductor of radius b. For the present symmetrical field with circular magnetic lines, we have, a priori, the following relation between H_φ and i_z at any r.

$$(2)\quad H_\varphi = \frac{2i_z}{cr}$$

Replacing H_φ in (1) by the above relation, we have:

$$(1)_a\quad E_z = \frac{2i\omega\mu}{c^2k^2}\frac{1}{r}\frac{\partial i_z}{\partial r}$$

$$(1)_b\quad E_r = \frac{-2i\omega\mu}{c^2k^2}\frac{1}{r}\frac{\partial i_z}{\partial z}$$

$$(1)_c\quad \frac{\partial E_r}{\partial z} - \frac{\partial E_r}{\partial r} = \frac{2i\omega\mu}{c^2}\frac{i_z}{r}$$

* A time factor $e^{-i\omega t}$ is assumed. Gaussian units are used throughout.

From these, we can obtain explicit expressions between the voltage V_{ba} and the current I_z. Integrating $(1)_b$ from $r = b$ to $r = a$, since $i_z = I_z$ remains constant for $b \leq r \leq a$ upon neglecting the current in the air medium, we have

$$V_{ba} = \int_b^a E_r\, dr = \frac{-2i\omega\mu_1}{c^2 k_1^2} \cdot \log\left(\frac{a}{b}\right) \cdot \frac{\partial I_z}{\partial z}$$

or

$$\frac{\partial I_z}{\partial z} = \frac{ic^2 k_1^2}{2\omega\mu_1 \log \frac{a}{b}} V_{ba} = -Y^* V_{ba} \tag{3}$$

This gives immediately the definition of the shunt admittance ($Y = G + i\omega C$) for the concentric transmission system in the conventional sense. The shunt conductance and the shunt capacitance per unit length are

$$G = \frac{2\pi\sigma_1}{\log \frac{a}{b}}, \quad \text{and} \quad C = \frac{\epsilon_1}{2 \log \frac{a}{b}} \tag{4}$$

respectively. Similarly, integration of $(1)_e$ yields

$$\frac{\partial}{\partial z}\int_b^a E_r\, dr - \int_b^a \frac{\partial E_z}{\partial r}\, dr = \frac{2i\omega\mu_1}{c^2} \cdot \log\frac{a}{b} \cdot I_z$$

or

$$\frac{\partial V_{ba}}{\partial z} = \frac{2i\omega\mu_1}{c^2} \cdot \log\frac{a}{b} \cdot I_z + [E_z(a) - E_z(b)] = -Z^* I_z \tag{5}$$

Note on Y^* and Z^*: The star sign is used here to indicate the complex conjugate quantities of Y and Z. It is necessary because a time factor $e^{-i\omega t}$ is used.

Equations (3) and (5) are formally analogous to the partial differential equations of the conventional transmission lines. In eq. (3), the circuit constant Y is completely determined, while in eq. (5) we have still to search for some explicit expressions relating the electric field intensity E_z at $r = a$ $[E_z(a)]$ and $r = b$ $[E_z(b)]$ with the current I_z. This can be obtained from Maxwell's field equations (1). It can be shown that the solutions of the field intensities for the different regions in series expansion form are:

(i) for $0 \leq r \leq b$, inside the central conductor

$$\begin{cases} E_z = Re.\left[\sum_\nu e^{i(\lambda_\nu z - \omega t)} \cdot A_\nu J_0(r\sqrt{k_2^2 - \lambda_\nu^2})\right] \\ E_r = Re.\left[\sum_\nu e^{i(\lambda_\nu z - \omega t)} \cdot \frac{-i\lambda_\nu}{\sqrt{k_2^2 - \lambda_\nu^2}} A_\nu J_1(r\sqrt{k_2^2 - \lambda_\nu^2})\right] \\ H_\varphi = Re.\left[\sum_\nu e^{i(\lambda_\nu z - \omega t)} \cdot \frac{ck_2^2}{i\omega\mu_2\sqrt{k_2^2 - \lambda_\nu^2}} A_\nu J_1(r\sqrt{k_2^2 - \lambda_\nu^2})\right] \end{cases} \tag{6}$$

(ii) for $b < r < a$, in the dielectric air medium

$$(7)\quad \begin{cases} E_z = Re.\left[\sum_\nu e^{i(\lambda_\nu z-\omega t)}\cdot\{B_\nu J_0(r\sqrt{k_1^2-\lambda_\nu^2}) + C_\nu H_0^{(1)}(r\sqrt{k_1^2-\lambda_\nu^2})\}\right] \\ E_r = Re.\left[\sum_\nu e^{i(\lambda_\nu z-\omega t)}\cdot\frac{-i\lambda_\nu}{\sqrt{k_1^2-\lambda_\nu^2}} \cdot\{B_\nu J_1(r\sqrt{k_1^2-\lambda_\nu^2}) + C_\nu H_1^{(1)}(r\sqrt{k_1^2-\lambda_\nu^2})\}\right] \\ H_\varphi = Re.\left[\sum_\nu e^{i(\lambda_\nu z-\omega t)}\cdot\frac{ck_1^2}{i\omega\mu_1\sqrt{k_1^2-\lambda_\nu^2}} \cdot\{B_\nu J_1(r\sqrt{k_1^2-\lambda_\nu^2}) + C_\nu H_1^{(1)}(r\sqrt{k_1^2-\lambda_\nu^2})\}\right] \end{cases}$$

(iii) for $a \leq r < \infty$, inside the outer conductor

$$(8)\quad \begin{cases} E_z = Re.\left[\sum_\nu e^{i(\lambda_\nu z-\omega t)}\cdot D_\nu H_0^{(1)}(r\sqrt{k_2^2-\lambda_\nu^2})\right] \\ E_r = Re.\left[\sum_\nu e^{i(\lambda_\nu z-\omega t)}\cdot\frac{-i\lambda_\nu}{\sqrt{k_2^2-\lambda_\nu^2}} D_\nu H_1^{(1)}(r\sqrt{k_2^2-\lambda_\nu^2})\right. \\ H_\varphi = Re.\left[\sum_\nu e^{i(\lambda_\nu z-\omega t)}\cdot\frac{ck_2^2}{i\omega\mu_2\sqrt{k_2^2-\lambda_\nu^2}} D_\nu H_1^{(1)}(r\sqrt{k_2^2-\lambda_\nu^2})\right] \end{cases}$$

Here the λ_ν's are the eigenvalues (complex propagation constants) corresponding to the different modes of propagation and are to be determined from the boundary conditions at $r = a$ and at $r = b$. That is: the tangential components of the electric and the magnetic field intensity must be continuous on these boundaries. The resulting transcendental equation for λ_ν is very complicated and its solution is prohibitive if not just impossible. Fortunately, for metal conductors of high conductivity, no explicit determination of λ_ν is necessary. In fact, we need only consider equations (6) and (8) to find $E_z(b)$ and $E_z(a)$ in terms of I_z. This can be accomplished by the following scheme.

The A_ν's can be expressed in terms of $I_z = \sum_\nu I_{z\nu}$, with $I_{z\nu}$'s as the current components corresponding to the different modes of propagation. By integrating H_φ in (6) around a circle with radius $r = b$, there results

$$(9)\quad \int_0^{2\pi} H_\varphi b\,dy = 2\pi b\sum_\nu e^{i(\lambda_\nu z-\omega t)}\cdot\frac{ck_2^2}{i\omega\mu_2\sqrt{k_2^2-\lambda_\nu^2}} A_\nu J_1(b\sqrt{k_2^2-\lambda_\nu^2}) = \frac{4\pi I_z}{c} = \frac{4\pi}{c}\sum_\nu I_{z\nu}$$

The above indicated decomposition of I_z into its "eigenmodes" of propagation, although later found unnecessary for the present development, is, however, very suggestive in some physical respects. It shows the intrinsic property of propagation with the corresponding eigenvalues λ_ν's and also the basic difference

between a concentric system and a hollow guide system. From (9) we determine the A_ν's as follows:

$$A_\nu = \frac{4\pi}{c} I_{z\nu} \cdot \frac{i\omega\mu_2\sqrt{k_2^2 - \lambda_\nu^2}}{2\pi b \cdot ck_2^2 e^{i(\lambda_\nu z - \omega t)}} \cdot \frac{1}{J_1(b\sqrt{k_2^2 - \lambda_\nu^2})}$$

Substituting this into the expression for E_z in (6), we have

$$E_z(b) = \sum_\nu \frac{i\omega\mu_2\sqrt{k_2^2 - \lambda_\nu^2}}{ck_2^2} \cdot \frac{J_0(b\sqrt{k_2^2 - \lambda_\nu^2})}{J_1(b\sqrt{k_2^2 - \lambda_\nu^2})} \cdot \frac{2}{bc} I_{z\nu}$$

(10)

$$\approx \frac{2i\omega\mu_2}{bc^2 k_2} \frac{J_0(bk_2)}{J_1(bk_2)} \sum_\nu I_{z\nu} = \frac{2i\omega\mu_2}{bc^2 k_2} \frac{J_0(bk_2)}{J_1(bk_2)} I_z$$

Thus it is seen that, for $k_2^2 \gg \lambda_\nu^2$, the "eigen-modes" of I_z finally recombine. (10) is therefore the required expression for $E_z(b)$ to be used in (5).

Following the same procedure, the D_ν's can be determined by integrating H_φ in (8) around a circle of radius $r = a$

$$\int_0^{2\pi} H_\varphi a \, d\varphi = 2\pi a \sum_\nu e^{i(\lambda_\nu z - \omega t)} \cdot \frac{ck_2^2}{i\omega\mu_2\sqrt{k_2^2 - \lambda_\nu^2}} \cdot D_\nu H_1^{(1)}(a\sqrt{k_2^2 - \lambda_\nu^2}) \tag{11}$$

Substituting these D_ν's into the expression for E_z in (8), we obtain

$$E_z(a) = \sum_\nu \frac{i\omega\mu_2\sqrt{k_2^2 - \lambda_\nu^2}}{ck_2^2} \cdot \frac{H_0^{(1)}(a\sqrt{k_2^2 - \lambda_\nu^2})}{H_1^{(1)}(a\sqrt{k_2^2 - \lambda_\nu^2})} \cdot \frac{-2}{ac} I_{z\nu}$$

(12)

$$\approx \frac{-2i\omega\mu_2}{ac^2 k_2} \frac{H_0^{(1)}(ak_2)}{H_1^{(1)}(ak_2)} \sum_\nu I_{z\nu} = \frac{-2i\omega\mu_2}{ac^2 k_2^2} \frac{H_0^{(1)}(ak_2)}{H_1^{(1)}(ak_2)} I_z$$

This is therefore the desired relation for $E_z(a)$ to be used in (5).

Before substituting (10) and (12) into (5), we can justifiably make a further simplification due to the very high conductivity of the metal conductors. From the asymptotic expansions of cylindrical functions (4), we have

$$\operatorname*{Lim}_{\sigma_2 \to \infty} \frac{J_0(bk_2)}{J_1(bk_2)} = \operatorname*{Lim}_{\sigma_2 \to \infty} \frac{J_0(b\sqrt{4\pi\omega\mu_2\sigma_2}\sqrt{i})}{J_1(b\sqrt{4\pi\omega\mu_2\sigma_2}\sqrt{i})} = -i$$

$$\operatorname*{Lim}_{\sigma_2 \to \infty} \frac{H_0^{(1)}(ak_2)}{H_1^{(1)}(ak_2)} = \operatorname*{Lim}_{\sigma_2 \to \infty} \frac{H_0^{(1)}(a\sqrt{4\pi\omega\mu_2\sigma_2}\sqrt{i})}{H_1^{(1)}(a\sqrt{4\pi\omega\mu_2\sigma_2}\sqrt{i})} = i$$

Then $E_z(a)$ and $E_z(b)$ become

$$E_z(a) = \frac{2\omega\mu_2}{ac^2 k_2} I_z = \frac{\omega\mu_2(1-i)}{ac\sqrt{2\pi\omega\mu_2\sigma_2}} I_z \tag{12}_a$$

$$E_z(b) = \frac{2\omega\mu_2}{bc^2 k_2} I_z = \frac{\omega\mu_2(1-i)}{bc\sqrt{2\pi\omega\mu_2\sigma_2}} I_z \tag{10}_a$$

Substituting these into (5), we obtain

$$\frac{\partial V_{ba}}{\partial z} = \left[\frac{2i\omega\mu_1}{c^2} \log\frac{a}{b} + \frac{\omega\mu_2(1-i)}{c\sqrt{2\pi\omega\mu_2\sigma_2}}\left(\frac{1}{a} - \frac{1}{b}\right)\right] I_z = -Z^* I_z \tag{13}$$

where

$$Z = R + i\omega L, \qquad R = \frac{\omega\mu_2}{c\sqrt{2\pi\omega\mu_2\sigma_2}}\left(\frac{a}{b} - \frac{1}{a}\right)$$

and

$$L = \frac{2\mu_1}{c^2}\log\frac{a}{b} + \frac{\mu_2}{c\sqrt{2\pi\omega\mu^2\sigma_2}}\left(\frac{1}{b} - \frac{1}{a}\right) \tag{14}$$

This is therefore the explicit definition of the series impedance of the concentric transmission system. R and L are the series resistance and the series inductance respectively.

The simple results obtained above for the definitions of the circuit constants R, G, C, L for a concentric system may prove very helpful for numerical computations. After these explicit determinations, we may now proceed to find the propagation characteristics in the manner used for conventional transmission circuits.

From (5) and (13), we have immediately the usual partial differential equations for the current and the voltage:†

$$(15) \qquad \frac{\partial^2 I_z}{\partial z^2} = ZYI_z = -\lambda^2 I_z; \qquad \frac{\partial^2 V_{bz}}{\partial z^2} = ZYV_{bz} = -\lambda^2 V_{bz}$$

The solutions are

$$(16) \qquad I_z = A_1 e^{-\lambda z} + A_2 e^{\lambda z}; \qquad V_{bz} = -A_1\sqrt{\frac{Z}{Y}}\cdot e^{-\lambda z} + A_2\sqrt{\frac{Z}{Y}}\cdot e^{\lambda z}$$

where the complex propagation constant λ, with a real component β and an imaginary component α, is defined as

$$(17) \qquad \lambda = \beta + i\alpha = i\sqrt{ZY} = i\sqrt{(R + i\omega L)(G + i\omega C)}$$

If the distance z is measured from the transmitter end, the first term in the expressions for I_z and V_{bz} in (16) represents the reflecting wave and the second term, the progressive wave. The constant A_1 would be identically equal to zero for an infinitely long line or, for a line perfectly matched at the receiving end. In such cases, equations (16) reduce to

$$(18) \qquad I_z = A_2 e^{\lambda z}; \qquad V_{bz} = A_2\sqrt{Z/Y}\, e^{\lambda z},$$

where A_2 is then the current strength at the transmitting end ($z = 0$). The "characteristic impedance" of the concentric transmission system, defined by the following relation

$$Z_0 \equiv \sqrt{Z/Y},$$

† In the following we shall neglect the effect of the time factor $e^{-i\omega t}$.

is the impedance which should be connected at the receiving end so that no reflection occurs. The phase constant β and the attenuation constant α can then be determined from relation (17). They are:

$$(19)\qquad \begin{cases} \beta^2 = \tfrac{1}{2}[(\omega^2 LC - RG) + \sqrt{(\omega^2 LC - RG)^2 + (R\omega C + G\omega L)^2}] \\ \alpha^2 = \tfrac{1}{2}[-(\omega^2 LC - RG) + \sqrt{(\omega^2 LC - RG)^2 + (R\omega C + G\omega L)^2}] \end{cases}$$

These are therefore rather general formulae for computation of the propagation constants, which must both be real positive quantities by definition. Now if we use the approximations: (a) $\sigma_1 \to 0$, $G \to 0$; and (b) $\sigma_2 \to$ very large, $R \to$ very small but finite, as stated at the beginning of this article, the attenuation constant and the phase constant become

$$(20)\qquad \alpha = \frac{R}{2}\sqrt{\frac{C}{L}} \quad \text{and} \quad \beta = \omega\sqrt{LC}$$

respectively. Substituting from (4) and (14) the expressions for R, L, and C, and neglecting the second term for L, we obtain

$$(21)\qquad \alpha = \frac{1}{4}\sqrt{\frac{\mu_2 \epsilon_1}{\mu_1 \sigma_2}}\left(\frac{1}{b} - \frac{1}{a}\right)\frac{f^{1/2}}{\log \frac{a}{b}} \quad \text{and} \quad (22)\qquad \beta = \frac{\omega\sqrt{\epsilon_1 \mu_1}}{c}$$

The phase velocity of propagation is thus essentially equal to that in free space

$$(23)\qquad v_p = \frac{\omega}{\beta} = \frac{c}{\sqrt{\epsilon_1 \mu_1}}$$

The above simple expression for α is suitable for numerical computation and first approximate discussions. The more accurate value has to be calculated from the more complicated relations (19), (4) and (14).

The "characteristic impedance" $Z_0 = \sqrt{(R + i\omega L)/(G + i\omega C)}$, for very small R and G, is approximately a pure resistance.

$$(24)\qquad Z_0 \approx \frac{2}{c}\sqrt{\frac{\mu_1}{\epsilon_1}}\log\frac{a}{b} = R_0$$

If the receiving end is terminated by this resistance (R_0) together with a small variable compensating reactive component, nearly perfect matching can be realized without much difficulty.

The field configuration in the dielectric air medium is automatically stabilized by the currents flowing forward and backward in the central and outer conductors, the magnetic intensity being circular and the electric intensity radial. The operation of the concentric system is therefore essentially a stable one.

Conclusion

Now we are in a position to make a comparison of the relative merits of a concentric transmission line and a hollow tube guide. They are summarized under the following items:

(1) Match of the transmission system to the transmitter.

By a perfect match, we mean that the energy is completely transmitted or radiated. This can be realized for both systems although in quite different physical sense. The hollow tube guide is characterized by the peculiar resonance phenomenon, especially for a linear antenna.

(2) Match of the receiver to the transmission system.

For a concentric system, this constitutes an identical problem as item (1). For a hollow guide system, no perfect match at the receiving end is conceivable at the present stage of knowledge.

(3) Reflection and standing wave.

For a concentric system, these do not exist because of easy matching at both ends. For a hollow guide system, they are unavoidable. This is the source of interference between the transmitting and the receiving devices and also the cause of distortion of the composition of the original signal.

(4) Efficiency of reception.

From above discussions, this is practically one hundred per cent for a concentric system and very poor for a hollow guide.

(5) Stability of field configuration and energy distribution.

The instability of a hollow guide with respect to field configuration and energy distribution is a serious nuisance toward application. For a concentric system, this constitutes no question.

(6) Attenuation constant.

For a hollow guide, considering only the first mode of propagation, the attenuation constant is

$$\alpha_1 = \frac{1}{2a}\sqrt{\frac{\mu_2 \epsilon_1}{\mu_1 \sigma_2}}\,\frac{f^{1/2}}{\sqrt{1-(f_1/f)^2}}$$

That for a concentric system is given by eq. (21). With the same inner radius, a, of the outer hollow conductor for both systems, their ratio is

$$q = \frac{\alpha}{\alpha_1} = \frac{\frac{a}{b}-1}{2\log\frac{a}{b}}\sqrt{1-\left(\frac{f_1}{f}\right)^2} \approx \frac{\frac{a}{b}-1}{2\log\frac{a}{b}} \qquad \text{for } f \gg f_1. \tag{25}$$

This varies approximately from 0.5 to 2.0 for the probable range of a/b from 1.5 to 10, with the frequency f near the point of minimum attenuation for the hollow guide. It is expected that the attenuation constant α for the concentric system would be somewhat increased due to the insertion of regular separating insulators to keep the central conductor in position. Consequently, in order to keep down the unavoidable increase of attenuation, it is recommended that only high quality, low loss materials should be used for the insulators. If this can be realized, a concentric system compares favorably with a hollow guide so far as attenuation is concerned.

(7) Future prospect.

From what has been discussed before, it is evident that, unless some fundamentally new schemes of reception and of stabilizing the field configuration can be achieved for a hollow guide, it will remain in an inferior position as compared with a concentric system for long distance transmission.

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PRIME DIVISORS AND ALMOST PERIODICITY

By Aurel Wintner

1. The following considerations deal with the pair of classical arithmetical functions usually denoted by $\nu(n)$ and $\tilde{\omega}(n)$, where $n = 1, 2, \cdots$. The first of these functions represents the number of the distinct prime divisors, the second the number of all prime divisors, of the variable integer n. Thus

$$\nu(n) = \sum_{p \mid n} 1, \qquad \tilde{\omega}(n) = \sum_{p^k \mid n} 1, \tag{1}$$

where p and k denote primes and positive integers respectively.

It is a general experience that the functions $\nu(n)$ and $\tilde{\omega}(n)$, though rather "wobbly" both in themselves and with reference to one another, show certain tendencies toward a *common* asymptotic behavior. For instance, if m is any fixed positive integer, then, as apparently observed already by Gauss,[1] the Prime Number Theorem is equivalent to the statement that the number of those integers not exceeding x which satisfy the restriction $\rho(n) = m$ is[2]

$$\sim \frac{(\lg\lg x)^{m-1}}{(m-1)!\lg x} \quad \text{as } x \to \infty, \tag{2}$$

where ρ denotes *either* of the functions ν, $\tilde{\omega}$. Thus the occurrence of *multiple* prime divisors represents an "abnormal" contingency, having a negligible asymptotic effect on problems of not too sensitive a type.

The purpose of the present note is to exhibit the analytical reasons for this general experience. The resulting explanation, based on a Fourier analysis of the deviation

$$\delta(n) = \tilde{\omega}(n) - \nu(n), \tag{3}$$

turns out to be explicit enough to supply a precise delimitation of some of the situations which are not "too sensitive."

Needless to say,

$$\delta(n) \geqq 0, \tag{4}$$

where the sign of equality holds if and only if n is square-free; cf. (1), (3).

2. It will be shown that the deviation of the two functions (1) is so regular and so small on the average, that the (non-negative, integral-valued) function (3) of n has an asymptotic distribution function and that the latter possesses momenta of arbitrarily high order. Furthermore, the "wobbly" behavior of the deviation turns out to be so regular that (3) not only possesses an asymptotic distribution function but is almost periodic (B^2) and, as a matter of fact, almost

[1] Cf. Enc. der math. Wiss., II C 8, (H. Bohr and H. Cramér), p. 828.

[2] For a statistical interpretation of these asymptotic relations in terms of Poisson's law of independent rare events, cf. A. Wintner, Statistics and prime numbers, Nature, vol. 147 (1941), pp. 208–209.

periodic (B^∞), that is, almost periodic (B^λ) for every λ. It is clear from (1) that (3), being an unbounded function of n, cannot be uniformly almost periodic. Actually, (3) cannot even be (B)-equivalent to a uniformly almost periodic function. In fact, the asymptotic distribution function of (3) turns out to have jumps at arbitrarily large integers.

The Fourier series of (3) appears in terms of the Ramanujan function[3] usually denoted by $c_m(n)$ and representing the sum of the n-th powers of the $\phi(m)$ roots of the m-th cyclotomic equation; so that

$$(5) \qquad c_m(n) = \sum_{(j,m)=1} \exp(2\pi ijn/m) = \sum_{(j,m)=1} \cos(2\pi jn/m),$$

where the summation index j runs through the $\phi(m)$ residue classes (mod m) relatively prime to m. Actually, these trigonometric functions of n will occur only when m is a prime power.

Except for the mean-value

$$(7) \qquad M(f) = \lim_{n\to\infty} \frac{f(1) + \cdots + f(n)}{n},$$

which for $f = \delta$ turns out to be

$$(8) \qquad M(\delta) = \sum_p \frac{1}{p(p-1)},$$

the Fourier coefficients in the harmonic analysis of (3) will simply be expressible in terms of Euler's ϕ-function. In fact, the Fourier series (B) of the almost periodic function (3) of n will be shown to be

$$(9) \qquad \delta(n) \sim M(\delta) + \sum_{k=1}^{\infty} \sum_p \frac{c_{p^k}(n)}{\phi(p^{k'})}, \qquad \text{where } k' = \begin{cases} k & \text{if } k > 1, \\ 2 & \text{if } k = 1. \end{cases}$$

[3] S. Ramanujan, Collected Papers, 1927, p. 180.

It may be mentioned that Ramanujan's identities imply for the derivative of the Riemann zeta function the curious representation

$$(6) \qquad -\zeta'(s) = \sum_{n=1}^{\infty} \sigma_{1-s}(n),$$

where $\sigma_z(n)$ denotes the sum of the z-th powers of the divisors of n (it being understood that k_z is defined to be $\exp(z \lg k)$ in terms of the *real* logarithm of k). In fact, Ramanujan proves (loc. cit., p. 185) that

$$\sigma_{-s}(n) = \sum_{m=1}^{\infty} c_m(n)/m^{s+1}, \qquad (\mathbf{R}(s) > 0).$$

On writing here s for $s + 1$ and combining the result with the relation

$$\Lambda(n) = \sum_{l=1}^{\infty} c_n(l)/l, \quad \text{where} \quad \sum_{n<x} \Lambda(n) = \sum_{p^k<x} \lg p,$$

(loc. cit., p. 199), one sees after straightforward rearrangements, that the desired representation of $\zeta'(s)$ is equivalent to the classical Dirichlet expansion

$$-\zeta'(s)/\zeta(s) = \sum_{n=1}^{\infty} \Lambda(n)/n^s, \qquad (\mathbf{R}(s) > 1).$$

3. By an additive function $f(n)$ is meant a sequence defined for every positive integer n in such a way that

(10) $f(n_1n_2) = f(n_1) + f(n_2)$ whenever $(n_1, n_2) = 1$ (hence, $f(1) = 0$).

Since (10) is equivalent to

$$f(n) = \sum_{p^k \mid n} f(p^k), \qquad p^{k+1} \nmid n, \tag{11}$$

an additive $f(n)$ is uniquely defined by an arbitrary allotment of its generating numbers $f(p^k)$, where p and k run through all primes and all positive integers respectively.

It is clear from (1) that (11) is satisfied by both $f = \nu$ and $f = \tilde{\omega}$, and that

$$\nu(p^k) = 1, \qquad \tilde{\omega}(p^k) = k.$$

It follows, therefore, from (3) that (11) is satisfied by $f = \delta$, and that

$$\delta(p^k) = k - 1. \tag{12}$$

It will be rather essential that, as a consequence of (12),

$$\delta(p) = 0. \tag{12 bis}$$

It was recently shown[4] that an additive function $f(n)$ is almost periodic (B^2) if and only if both series

$$\sum_p \frac{f(p)}{p}, \qquad \sum_{k=1}^{\infty} \sum_p \frac{|f(p^k)|^2}{p^k}$$

are convergent. It is clear from (12) that, if $f = \delta$, the first of these two series reduces to $0 + 0 + \cdots$ and the second to

$$\sum_{k=1}^{\infty} \sum_p \frac{(k-1)^2}{p^k} < \sum_{k=2}^{\infty} k^2 \sum_p \frac{1}{p^k} < \sum_{k=2}^{\infty} k^2 \frac{\text{const.}}{2^k} < \infty.$$

Hence, the function (3) of n is almost periodic (B^2).

4. It was also shown loc. cit.[4] that, if an additive function $f(n)$ is almost periodic (B^2), its Fourier series reduces, in terms of the Ramanujan sums (5) having prime powers as subscripts, to the expansion

$$f(n) \sim M(f) + \sum_{k=1}^{\infty} \sum_p a_{p^k}(f) c_{p^k}(n),$$

where the mean-value (7) is given by

$$M(f) = \sum_p \frac{p-1}{p} \sum_{k=1}^{\infty} \frac{f(p^k)}{p^k}$$

[4] P. Erdös and A. Wintner, Additive functions and almost periodicity, American Journal of Mathematics, vol. 61 (1939), pp. 713–721.

and every other Fourier constant by

$$a_{p^k}(f) = \sum_{j=k}^{\infty} \frac{f(p^j) - f(p^{j-1})}{p^j}.$$

Thus, if $k > 1$, then, from (12),

$$a_{p^k}(\delta) = \sum_{j=k}^{\infty} \frac{(j-1) - (j-2)}{p^j} = \frac{1/p^k}{1 - 1/p} = \frac{1}{p^k - p^{k-1}} = \frac{1}{\phi(p^k)}.$$

If, on the other hand, $k = 1$, then, since $f(1) = 0$ for every additive function,

$$a_p(\delta) = \sum_{j=2}^{\infty} \frac{(j-1) - (j-2)}{p^j} = \frac{1/p^2}{1 - 1/p} = \frac{1}{p^2 - p} = \frac{1}{\phi(p^2)}.$$

This proves (9).

The series representation of the mean-value reduces to

$$M(\delta) = \sum_p \frac{p-1}{p} \sum_{k=1}^{\infty} \frac{\delta(p^k)}{p^k} = \sum_p \frac{p-1}{p} \sum_{k=1}^{\infty} \frac{k-1}{p^k},$$

by (12). Hence, (8) follows from

$$\sum_{k=1}^{\infty} ka^k = \frac{a}{(1-a)^2} \quad \text{and} \quad \sum_{k=1}^{\infty} a^k = \frac{a}{1-a}, \qquad \text{where } a = \frac{1}{p}.$$

5. For every fixed $m \geqq 1$, the almost periodicity (B^m) of an additive function can be characterized by the simultaneous convergence of the four series[5]

$$\sum_p \frac{f(p)}{p}, \qquad \sum_p \frac{|f^+(p)|^2}{p}, \qquad \sum_{k=2}^{\infty} \sum_p \frac{|f(p^k)|^m}{p^k}, \qquad \sum_{|f(p)| \geqq 1} \frac{|f(p)|^m}{p},$$

where $f^+(n)$ is defined by placing, for $y = f(n)$,

$$y^+ = \begin{cases} y & \text{if } |y| < 1, \\ 0 & \text{if } |y| \geqq 1. \end{cases}$$

It is clear from (12) that the first, second and fourth of these series reduce to $0 + 0 + \cdots$, and that the third is

$$\sum_{k=2}^{\infty} \sum_p \frac{(k-1)^m}{p^k} < \sum_{k=2}^{\infty} k^m \sum_p \frac{1}{p^k} < \sum_{k=2}^{\infty} k^m \frac{\text{const.}}{2^k} < \infty,$$

where m is arbitrarily fixed. Hence, the function (3) of n is almost periodic (B^m) for arbitrarily large values of the index m.

[5] P. Hartman and A. Wintner, On the almost periodicity of additive number-theoretical functions, American Journal of Mathematics, vol. 62 (1940), pp. 753–758 (more particularly, p. 758).

In particular, there exists a finite mean-value (7) for $f = \delta^m$, where $m = 1, 2, \cdots$. This implies[6] that, if $\sigma(x)$, $-\infty < x < \infty$, denotes the asymptotic distribution function of the almost periodic function (3), then

$$M(\delta^m) = \int_{-\infty}^{\infty} x^m \, d\sigma(x) = \int_{0}^{\infty} x^m \, d\sigma(x)$$

for every m. In particular, σ possesses momenta of arbitrarily high order.

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[6] Cf. B. Jessen and A. Wintner, Distribution functions and the Riemann zeta function, Transactions of the American Mathematical Society, vol. 38 (1935), pp. 48–88 (more particularly, p. 76).

QUADRATIC CONVOLUTION EQUATIONS[1]

By R. H. Cameron

1. The equations and their discriminants. This paper deals with the equation

$$(1)\qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x-t-u)\,dF(u)\,dA(t) + 2\int_{-\infty}^{\infty} F(x-t)\,dB(t) + C(x) = 0,$$

in which $A(x)$, $B(x)$, $C(x)$ are right continuous functions of bounded variation on $-\infty < x < +\infty$, and we seek for solutions of the same type. The integrals are understood to be Lebesgue-Stieltjes (Radon) integrals.

We also discuss the important special cases

$$(2)\qquad \begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-t-u)f(u)P(t)\,dt\,du + p_0\int_{-\infty}^{\infty} f(x-t)f(t)\,dt \\ + 2\int_{-\infty}^{\infty} f(x-t)Q(t)\,dt + 2q_0 f(x) + R(x) = 0\end{aligned}$$

and

$$(3)\qquad \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} a_{n-j-k}f_j f_k + 2\sum_{k=-\infty}^{\infty} b_{n-k}f_k + c_n = 0 \qquad (n = 0, \pm 1, \pm 2, \cdots)$$

and the system

$$(3a)\qquad \begin{cases}\displaystyle\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} a_{n-j-k}(f_j f_k + g_j g_{k+1}) + 2\sum_{k=-\infty}^{\infty} b_{n-k}f_k + c_n = 0 \\ \displaystyle\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} a_{n-j-k}f_j g_k + \sum_{k=-\infty}^{\infty} b_{n-k}g_k = 0 \qquad (n = 0, \pm 1, \pm 2, \cdots).\end{cases}$$

In (2) the coefficients $P(x)$, $Q(x)$, $R(x)$ are of class L_1 and we seek solutions $f(x)$ of class L_1. In (3) the sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ are of class l_1, (i.e., the sum of their absolute values converges), and we seek solutions of class l_1. The system (3a) will be studied in connection with (3), and its coefficients and solutions are also of class l_1.

In connection with these equations we define the discriminants

$$(4)\qquad D(x) = \int_{-\infty}^{\infty} [B(x-t)\,dB(t) - A(x-t)\,dC(t)]$$

for (1),

$$(5)\qquad S(x) = \int_{-\infty}^{\infty} [Q(x-t)Q(t) - P(x-t)R(t)]\,dt + 2q_0 Q(x) - p_0 R(x)$$

for (2), and

$$(6)\qquad t_n = \sum_{n=-\infty}^{\infty} (b_{n-k}b_k - a_{n-k}c_k)$$

for (3) and (3a).

[1] Presented to the Society September 3, 1941

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2. Non-singular functions of bounded variation. It is well known that every function of bounded variation can be expressed as the sum of three functions, of which one is a step function, the second is absolutely continuous, and the third is continuous and almost everywhere has a vanishing derivative. These three parts are called the discrete, absolutely continuous, and singular parts respectively, and they are uniquely determined up to an additive constant. We take the singular part as vanishing at some convenient point, say at $-\infty$, and if it vanishes everywhere, we call the function non-singular. Thus a non-singular function of bounded variation is a function of bounded variation which can be expressed as the sum of a step-function and an absolutely continuous function. We begin by stating theorems for non-singular functions in connection with equation (1). We take first real coefficients and then complex ones.

THEOREM I. *Let $A(x)$, $B(x)$, $C(x)$ be real non-singular right continuous functions of bounded variation such that $C(-\infty) = 0$ and the Fourier-Stieltjes transforms of $A(x)$ and the discriminant $D(x)$ defined in* (4) *are both bounded away from zero. Let $d_1 = D(+\infty) - D(-\infty)$, and let d_2 be the algebraic sum of the jumps of $D(x)$. Then if d_1 and d_2 have the same sign, the equation* (1) *has two and only two right continuous solutions $F(x)$ of bounded variation which vanish at $-\infty$, and both of these solutions are non-singular. Moreover, both are real if d_1 and d_2 are positive, while the two solutions are distinct complex conjugates if d_1 and d_2 are negative. If d_1 and d_2 have opposite signs, there is no solution of bounded variation.*

THEOREM IA. *Let $A(x)$, $B(x)$, $C(x)$ be complex non-singular right continuous functions of bounded variation such that $C(-\infty) = 0$ and the Fourier-Stieltjes transforms of $A(x)$ and $D(x)$ are bounded away from zero. Then the ratio of the Fourier-Stieltjes transform of the discrete part of $D(x)$ to the transform of $D(x)$ approaches unity at both ends of the real axis, and hence it winds about the origin an integral number of times (say n) as t varies from $-\infty$ to $+\infty$. Moreover if n is even, there are two and only two right continuous solutions of* (1) *of bounded variation which vanish at $-\infty$, and both are non-singular. If n is odd,* (1) *has no solutions of bounded variation.*

It is of course clear that the Fourier-Stieltjes transformation takes (1) into the quadratic equation

$$\alpha(t)[\phi(t)]^2 + 2\beta(t)\phi(t) + \gamma(t) = 0, \tag{7}$$

where α, β, γ, ϕ are the transforms of A, B, C, F. We thus have $\phi(t)$ given by the quadratic formula. Theorems I and I(a) give conditions under which this formal manipulation is justified.

If the hypotheses of Theorem I or Theorem I(a) are satisfied, the two solutions of (1) *are given (almost everywhere) by the formula*[2]

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-itx} - e^{-it\xi}}{-it} \left[\frac{-\int_{-\infty}^{\infty} e^{it\xi}\, dB(\xi) \pm \sqrt{\int_{-\infty}^{\infty} e^{it\xi}\, dD(\xi)}}{\int_{-\infty}^{\infty} e^{it\xi}\, dA(\xi)} \right] dt + \text{Const.} \tag{8}$$

[2] It is understood that the square roots of continuous functions are to be so chosen as to be continuous.

3. Proof of Theorems I and I(a). For the proof of Theorem I(a), let $D_1(x)$ be the discrete part of $D(x)$, and let the Fourier-Stieltjes transforms of $A(x)$, $B(x)$, $C(x)$, $D(x)$, $D_1(x)$ be $\alpha(t)$, $\beta(t)$, $\gamma(t)$, $\Delta(t)$, $\Delta_1(t)$. By hypothesis, $\alpha(t)$ and $\Delta(t)$ are bounded away from zero, and $\Delta_1(t)$ is also bounded away from zero since it is an almost periodic function whose difference from $\Delta(t)$ approaches zero as $t \to \pm\infty$. Thus the ratio $\Delta_1(t)/\Delta(t)$ approaches unity as $t \to \pm\infty$, and it winds n times around the origin as t goes from $-\infty$ to $+\infty$, n being an integer. Now by a theorem of Walther,[3] it follows that $\pm\sqrt{\Delta_1(t)}$ are both almost periodic functions, and from a theorem of Cameron and Wiener[4] that a particular continuous choice of $\sqrt{\Delta(t)}$ will be a Fourier-Stieltjes transform if it approaches a particular choice of $\sqrt{\Delta_1(t)}$ at both ends of the axis. Now if we choose $\sqrt{\Delta_1(t)}$ so that its ratio to $\sqrt{\Delta(t)}$ is unity at $-\infty$, this ratio will be $(-1)^n$ at $+\infty$, since $\sqrt{\Delta_1(t)}/\sqrt{\Delta(t)}$ winds $\frac{1}{2}n$ times around the origin. Thus $\sqrt{\Delta(t)}$ is a Fourier-Stieltjes transform if and only if n is even.

Now if n is even, the numerator and denominator of the bracketed expression in (8) are both Fourier-Stieltjes transforms of a function of B.V., and hence by a theorem of Wiener and Pitt,[5] the whole bracketed expression $f(t)$ is a Fourier-Stieltjes transform, and it is the transform of the $F(x)$ defined by (8). Then we have

$$f(t) = \frac{-\beta(t) \pm \sqrt{\Delta(t)}}{\alpha(t)} \tag{9}$$

and

$$\alpha(t)[f(t)]^2 + 2\beta(t)f(t) + \gamma(t) = 0. \tag{10}$$

But the transform of the left member of (1) is the left member of (10) and thus vanishes, so the left member of (1) must be equivalent to a constant, and since it approaches zero at $-\infty$ if $F(x)$ does, it must be equivalent to zero. Thus this $F(x)$ is a solution of (1).

Again, if $F(x)$ is a solution of (1), we obtain (10) by taking Fourier-Stieltjes transforms, so that

$$\Delta(t) = [\alpha(t)f(t) + \beta(t)]^2.$$

Thus $\Delta(t)$ is the square of a Fourier-Stieltjes transform, and n must be even. Moreover (9) and hence (8) must hold. We thus have no solution when n is odd, and just the two given by (8) when n is even. Consideration of the method of proof of the CW theorem will show that the two solutions of (1) given by (8) are non-singular, and complete the proof of Theorem I(a).

We can now prove Theorem I by applying Theorem I(a) to real functions $A(x)$, $B(x)$, $C(x)$. It is clear that $D(x)$ and $D_1(x)$ are also real, and that $\alpha(t)$, $\beta(t)$, $\gamma(t)$, $\Delta(t)$, $\Delta_1(t)$ satisfy equations of the form $\alpha(-t) = \overline{\alpha(t)}$. Now by definition, $d_1 = \Delta(0)$ and $d_2 = \Delta_1(0)$, so that $d_2/d_1 = \Delta_1(0)/\Delta(0)$. Since $\Delta_1(\infty)/\Delta(\infty) = 1$, it follows that arg $[\Delta_1(t)/\Delta(t)]$ increases by a whole number

[3] Algebraische Funktionen der fastperiodischen Funktionen, Monatshefte für Mathematik, vol. 40 (1933), pp. 444–457.

[4] Convergence Properties of Analytic Functions of Fourier-Stieltjes Transforms, Trans. Am. Math. Soc., Vol. 46, No. 1, pp. 97–104. This paper will hereafter be referred to as CW.

[5] On Absolutely Convergent Fourier-Stieltjes Transforms, Duke Math. J., vol. 4 (1938), pp. 420–436. This paper will be referred to as WP.

of revolutions or an odd number of half revolutions as t goes from 0 to $+\infty$ according as d_2/d_1 is positive or negative. But due to the conjugate symmetry of $\Delta_1(t)/\Delta(t)$, it follows that arg $[\Delta_1(t)/\Delta(t)]$ increases by the same amount in going from $-\infty$ to 0 as in going from 0 to $+\infty$. Thus n is even or odd according as d_2/d_1 is positive or negative, and by Theorem II(a), there are respectively two solutions or no solutions.

If d_2/d_1 is positive, so that the two solutions exist, we can determine whether or not they are real by determining whether or not $f(t)$ has conjugate symmetry. Since $\alpha(t)$, $\beta(t)$, $\Delta(t)$ have this symmetry, it follows from (9) that $f(t)$ will have it if and only if $\delta(t) = \sqrt{\Delta(t)}$ has it. But since $[\delta(-t)]^2 = [\overline{\delta(t)}]^2$, $\delta(-t) = \pm\overline{\delta(t)}$, and whichever sign is taken for $t = 0$ must be taken for all other t by continuity. But the positive sign will be taken if and only if $\delta(0)$ is real, and hence if and only if $d_1 > 0$. This completes the proof of Theorem I.

4. **Singular functions and special norms.** It is clear from the WP and CW theorems on which the proof of Theorem I(a) is based, that our hypothesis that $A(x)$, $B(x)$, $C(x)$ be non-singular is unnecessarily strong, and that all we need to do is to restrict the size of the singular parts of $A(x)$ and $D(x)$. We accordingly have

THEOREM I*. *Theorems I and I(a) remain true if we omit the restriction that the coefficients be non-singular, provided that we strengthen the hypothesis by including the additional conditions that for all t*

$$\int_{-\infty}^{\infty} |\, dA_2(x)\,| < \eta < \left| \int_{-\infty}^{\infty} e^{itx}\, dA_1(x) \right|$$

and

$$\int_{-\infty}^{\infty} |\, dD_2(x)\,| < \theta < \left| \int_{-\infty}^{\infty} e^{itx}\, dD_1(x) \right|$$

where A_1 and D_1 are the discrete parts and A_2 and D_2 the singular parts of A and D respectively. We must of course also omit from the conclusion of the theorem the statement that the solutions are non-singular. Moreover our conclusion that $\Delta_1(t)/\Delta(t) \to 1$ as $t \to \pm\infty$ must now be interpreted in the mean square sense, and the definition of n accordingly modified.

The CW theorem also deals with functions which satisfy stronger conditions than absolute convergence by introducing other norms than the total variation of the functions. The existence (finiteness) of these norms implies the finiteness of the total variation but not vice versa. It is thus possible to extend Theorem I still further by requiring that the coefficients be of finite norm and giving conditions which will produce solutions of finite norm. We will not give a formal statement of this theorem.

5. **Special Cases.** Certain special cases of theorems I and I(a) are as interesting as the general theorem itself. For instance, suppose that $I(x)$ is a function which is constant except for a unit jump at the origin and that

$$A(x) = p_0 I(x) + \int_{-\infty}^{x} P(t)\, dt, \qquad B(x) = q_0 I(x) + \int_{-\infty}^{x} Q(t)\, dt$$

and

$$C(x) = \int_{-\infty}^{x} R(t)\, dt,$$

where $P(x)$, $Q(x)$ and $R(x)$ belong to L_1. Then under certain conditions one solution of (1) will be of the form $F(x) = \int_{-\infty}^{x} f(t)\, dt$, where $f(t) \in L_1$. In terms of the new notation, equation (1) takes the form of equation (2) after differentiation of both sides. Equation (2) is then valid almost everywhere. Moreover it is clear that $D(x) = s_0 I(x) + \int_{-\infty}^{x} S(t)\, dt$, where $s_0 = q_0^2$ and $S(x)$ is given by (5). We thus obtain

THEOREM II. *Let p_0 and q_0 be real non-vanishing constants and let $P(x)$, $Q(x)$, $R(x)$ be real functions of class L_1 such that when $S(x)$ is defined by (5) the Fourier transform $\int_{-\infty}^{\infty} e^{itx} P(x)\, dx$ of $P(x)$ is never equal to $-p_0$ and the Fourier transform of $S(x)$ is never equal to $-q_0^2$. Then if $\int_{-\infty}^{\infty} S(x)\, dx > -q_0^2$ the equation (2) has one and only one solution $f(x)$ of class L_1, and this solution is real, while if $\int_{-\infty}^{\infty} S(x)\, dx < -q_0^2$, equation (2) has no (real or complex) solution of class L_1.*

THEOREM IIA. *Let p_0 and q_0 be (real or complex) non-vanishing constants and let $P(x)$, $Q(x)$, $R(x)$ be functions of class L_1, such that if $S(x)$ is defined by (5), the Fourier transform of $P(x)$ never equals $-p_0$ and the Fourier transform of $S(x)$ never equals $-q_0^2$. Then if the point in the complex plane representing $\int_{-\infty}^{\infty} e^{itx} S(x)\, dx$ winds around $-q_0^2$ an even number of times as t goes from $-\infty$ to $+\infty$, equation (2) has one and only one solution of class L_1, while if it winds an odd number of times around $-q_0^2$ equation (2) has no solution of class L_1.*

Theorems II and IIa follow immediately from I and Ia since integration takes (2) into (1) by means of the substitutions given at the beginning of this section, and one of the two solutions given by (8) has no jump at the origin (and hence is absolutely continuous). For the constant term of the expression under the radical in (8) is q_0^2, and hence the constant term for the radical is $\pm q_0$. This will cancel with the preceding term only if the upper sign is taken. Thus we have

The solution of (2) when the hypothesis of Theorem II or Theorem IIa is satisfied is[6]

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \left[\frac{-q_0 - \int_{-\infty}^{\infty} e^{it\xi} Q(\xi)\, d\xi + \sqrt{q_0^2 + \int_{-\infty}^{\infty} e^{it\xi} S(\xi)\, d\xi}}{p_0 + \int_{-\infty}^{\infty} e^{it\xi} P(\xi)\, d\xi} \right] dt.$$

[6] It is understood that the radical is to be so chosen that the square root is continuous and approaches q_0 at both ends of the axis.

Another interesting special case is obtained by requiring $A(x)$, $B(x)$, $C(x)$ to be step functions which have jumps a_n, b_n, c_n when $x = n$ ($n = 0, \pm 1, \pm 2, \cdots$). The condition that $A(x)$ etc. of B.V. becomes the condition that a_n etc. belong to l_1, and the Fourier-Stieltjes transforms become ordinary Fourier series of periodic functions of period 2π. But since the extraction of a square root may double the length of the period the solutions are step functions which may have jumps both at the integers and half integers. We will denote the jumps of $F(x)$ at integers n by f_n and at half integers $n - \frac{1}{2}$ by g_n. Thus equation (1) goes over into the system (3a) which reduces to (3) if there are no jumps at half integers, i.e., if the expression under the radical in (8) is the period 2π.

THEOREM III. *Let a_n, b_n, c_n ($n = 0, \pm 1, \pm 2, \cdots$) be real sequences of class l_1 such that $\sum_{n=-\infty}^{\infty} a_n e^{inx}$ is never zero, and such that $\sum_{n=-\infty}^{\infty} t_n e^{inx}$ (where t_n is defined by (6)) is never zero. Then equation (3a) has two and only two solutions of class l_1, and these solutions are real if $\sum_{n=-\infty}^{\infty} t_n > 0$ but are complex conjugates if $\sum_{n=-\infty}^{\infty} t_n < 0$. Moreover if $\sum_{n=-\infty}^{\infty} t_n$ and $\sum_{n=-\infty}^{\infty} (-1)^n t_n$ have the same signs, $g_n \equiv 0$ in the solutions of (3a) and hence (3) has two and only two solutions of class l_1, consisting of the f_n of (3a). On the other hand, if $\sum_{n=-\infty}^{\infty} t_n$ and $\sum_{n=-\infty}^{\infty} (-1)^n t_n$ have opposite signs equation (3) has no solutions of class l_1.*

THEOREM IIIA. *Let a_n, b_n, c_n ($n = 0, \pm 1, \pm 2, \cdots$) be sequences of class l_1 such that $\sum_{n=-\infty}^{\infty} a_n e^{inx}$ is never zero and such that $\sum_{n=-\infty}^{\infty} t_n e^{inx}$ (where t_n is defined by (6)) is never zero. Then equation (3a) has two and only two solutions of class l_1. Moreover if the point in the complex plane representing $\sum_{n=-\infty}^{\infty} t_n e^{inx}$ winds an even number of times around the origin as x goes from $-\pi$ to π, then $g_n \equiv 0$ in the solution of (3a), and hence (3) has two and only two solutions of class l_1 consisting of the f_n of (3a). On the other hand, if the point winds an odd number of times around the origin, then equation (3) has no solution of class l_1.*

These theorems follow immediately from Theorem I and the properties of periodic functions.

The solution of (3a) when the hypothesis of Theorem III or IIIa is satisfied is given by

$$f_k = \frac{1}{4\pi} \int_{-2\pi}^{2\pi} e^{-ikx} \left[\frac{-\sum_{n=-\infty}^{\infty} b_n e^{inx} \pm \sqrt{\sum_{n=-\infty}^{\infty} t_n e^{inx}}}{\sum_{n=-\infty}^{\infty} a_n e^{inx}} \right] dx,$$

$$g_k = \frac{1}{4\pi} \int_{-2\pi}^{2\pi} e^{-i(k-\frac{1}{2})x} \left[\frac{-\sum_{n=-\infty}^{\infty} b_n e^{inx} \pm \sqrt{\sum_{n=-\infty}^{\infty} t_n e^{inx}}}{\sum_{n=-\infty}^{\infty} a_n e^{inx}} \right] dx.$$

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SOME REMARKS CONCERNING THE COEFFICIENTS OF SCHLICHT FUNCTIONS

By D. C. Spencer

1. Suppose that s is a positive integer, and let F_s denote the family of functions $f_s(z)$ which are regular and schlicht for $|z| < 1$, and of the form

$$f_s(z) = z + c_{s+1}z^{s+1} + \cdots + c_{ns+1}z^{ns+1} + \cdots . \tag{1.1}$$

Let

$$f_s^{(0)}(z) = \frac{z}{(1 - z^s)^{2/s}} = z + a_{s+1}^{(0)} z^{s+1} + \cdots + a_{ns+1}^{(0)} z^{ns+1} + \cdots , \tag{1.2}$$

where

$$a_{ns+1}^{(0)} = \frac{1}{n!} \cdot \frac{2}{s} \cdot \left(\frac{2}{s} + 1\right) \cdots \left(\frac{2}{s} + n - 1\right). \tag{1.3}$$

It is well known that

$$|c_{s+1}| \leq a_{s+1}^{(0)} = \frac{2}{s}, \tag{1.4}$$

with equality only if $f_s = f_s^{(0)}$, and various authors have thus been led to consider the more general inequality

$$|c_{ns+1}| \leq a_{ns+1}^{(0)} \qquad (n = 1, 2, \cdots). \tag{1.5}$$

For $n = 2$ and $s = 1$, Löwner (4) has shown that (1.5) is true throughout F_1; for $n = 2$ and $s > 1$, Fekete and Szegö (2) have shown that there are functions of F_s for which it is false.[1]

My purpose here is twofold: (i) to construct explicitly,[2] for any $n > 1$, functions f_2 for which $|c_{2n+1}| > a_{2n+1}^{(0)} = 1$; (ii) to prove that when $s = 1$ analagous constructions necessarily fail to disprove (1.5) for any $n > 1$. In fact, if $n > 1$, infinitesimal deformations of $f_2^{(0)}$ increase $|a_{2n+1}^{(0)}|$, whereas infinitesimal deformations of $f_1^{(0)}$ depress $|a_n^{(0)}|$—a local maximum of $|c_n| = |c_n(f_1)|$ therefore occurs at $f_1 = f_1^{(0)}$. That $|c_4|$ has a local maximum at $f_1^{(0)}$ has been demonstrated by Joh (3), but his method does not extend to general n.

2. Let h denote a non-negative real number, η a complex number for which $|\eta| = 1$, and let $\zeta = \zeta(h, \eta, z)$ be determined from the equation

$$\eta\zeta + \frac{1}{\eta\zeta} = (1 + h)\left(\eta z + \frac{1}{\eta z}\right) + 2h. \tag{2.1}$$

[1] Moreover, they give the exact upper bound of $|c_{2s+1}|$ for $s = 2, 3, \cdots$.

[2] Fekete and Szegö establish the existence of functions f_s, $s > 1$, for which (1.5) (with $n = 2$) is false.

Let

$$\zeta_s = \zeta_s(h, \eta, z) = \{\zeta(h, \eta, z^s)\}^{1/s} = \sum_{n=0}^{\infty} b_{ns+1} z^{ns+1}. \tag{2.2}$$

Then plainly

$$b_1 = 1 - \frac{h}{s} \tag{2.3}$$

$$[b_{ns+1}]_{h=0} = \begin{cases} 1, & n = 0 \\ 0, & n > 0 \end{cases}. \tag{2.4}$$

Given $2k$ parameters (h_j, η_j), $j = 1, 2, \cdots, k$, we write

$$\zeta_s^{(j)}(z) = \zeta_s(h_j, \eta_j, z) = \sum_{n=0}^{\infty} b_{ns+1}^{(j)} z^{ns+1} \quad (j = 1, 2, \cdots, k), \tag{2.5}$$

and define (successively)

$$f_s^{(1)}(z) = \frac{1}{b_1}[f_s^{(0)}\{\zeta_s^{(1)}(z)\}], \cdots, f_s^{(k)}(z) = \frac{1}{b_1^{(k)}}[f_s^{(k-1)}\{\zeta_s^{(k)}(z)\}]. \tag{2.6}$$

For brevity we write

$$h^{(k)} = (h_1, h_2, \cdots, h_k); \qquad \eta^{(k)} = (\eta_1, \eta_2, \cdots, \eta_k); \tag{2.7}$$

and $h^{(k)} = \alpha$, for example, means that $h_1 = h_2 = \cdots = h_k = \alpha$. Then $f_s^{(k)}(z) = f_s^{(k)}(h^{(k)}, \eta^{(k)}, z)$, and

$$f_s^{(k)}(z) = z + a_{s+1}^{(k)} z^{s+1} + \cdots + a_{ns+1}^{(k)} z^{ns+1} + \cdots, \tag{2.8}$$

say. Moreover it is easy to see that

$$f_s^{(k)}(0, \eta^{(k)}, z) = f_s^{(k)}(h^{(k)}, -1, z) = f_s^{(0)}(z). \tag{2.9}$$

To avoid repetition, we suppose throughout that the integers s, n, j, and k satisfy the inequalities

$$s \geq 1, \qquad n \geq 1, \qquad 1 \leq j \leq k, \tag{2.10}$$

unless some additional restriction is explicitly stated.

3. The results of the paper are based on the following theorem:

THEOREM 1.

$$\left[\frac{\partial a_{ns+1}^{(k)}}{\partial h_j}\right]_{h^{(k)}=0} = -\frac{2}{s}\sum_{\mu=0}^{n-1}(\mu s + 1)a_{\mu s+1}^{(0)}\eta_j^{n-\mu} - n a_{ns+1}^{(0)}. \tag{3.1}$$

For the proof we require two easy lemmas:

LEMMA 1.

$$\left[\frac{\partial a_{ns+1}^{(k)}}{\partial h_j}\right]_{h^{(k)}=0} = \left[\frac{\partial a_{ns+1}^{(1)}}{\partial h_1}\right]_{h_1=0;\, \eta_1=\eta_j}. \tag{3.2}$$

LEMMA 2.

$$\left[\frac{\partial b_{ns+1}}{\partial h}\right]_{h=0} = -\frac{2}{s}\eta^n. \tag{3.3}$$

We prove Lemma 1 by induction, and suppose that (3.2) is true for $k = 1, 2, \cdots, \kappa - 1$. We have, writing $\lambda = \mu s + 1$,

$$\{\zeta_s^{(\kappa)}\}^{\mu s+1} = \sum_{n=\mu}^{\infty} \gamma_{ns+1}^{(\mu)} z^{ns+1}, \tag{3.4}$$

where

$$\gamma_{ns+1}^{(\mu)} = \sum_{\nu_1+\nu_2+\cdots+\nu_\lambda+\mu=n} b_{\nu_1 s+1}^{(\kappa)} b_{\nu_2 s+1}^{(\kappa)} \cdots b_{\nu_\lambda s+1}^{(\kappa)}; \tag{3.5}$$

and so (by (2.6))

$$\begin{aligned} f_s^{(\kappa)}(z) &= \frac{1}{b_1^{(\kappa)}}[f_s^{(\kappa-1)}\{\zeta_s^{(\kappa)}(z)\}] = \frac{1}{b_1^{(\kappa)}}\sum_{\mu=0}^{\infty} a_{\mu s+1}^{(\kappa-1)}\{\zeta_s^{(\kappa)}(z)\}^{\mu s+1} \\ &= \frac{1}{b_1^{(\kappa)}}\sum_{\mu=0}^{\infty} a_{\mu s+1}^{(\kappa-1)}\left\{\sum_{n=\mu}^{\infty}\gamma_{ns+1}^{(\mu)} z^{ns+1}\right\}. \end{aligned} \tag{3.6}$$

Hence

$$a_{ns+1}^{(\kappa)} = \frac{1}{b_1^{(\kappa)}}\cdot\sum_{\mu=0}^{n} a_{\mu s+1}^{(\kappa-1)}\cdot\gamma_{ns+1}^{(\mu)}. \tag{3.7}$$

By (2.9)

$$[a_{\mu s+1}^{(\kappa-1)}]_{h^{(\kappa-1)}=0} = a_{\mu s+1}^{(0)}; \tag{3.8}$$

therefore

$$\left[\frac{\partial a_{ns+1}^{(\kappa)}}{\partial h_\kappa}\right]_{h^{(\kappa)}=0} = \sum_{\mu=0}^{n} a_{\mu s+1}^{(0)}\left[\frac{\partial}{\partial h_\kappa}\left(\frac{\gamma_{ns+1}^{(\mu)}}{b_1^{(\kappa)}}\right)\right]_{h_\kappa=0} = \left[\frac{\partial a_{ns+1}^{(1)}}{\partial h_1}\right]_{h_1=0;\,\eta_1=\eta_\kappa}. \tag{3.9}$$

Finally, by (2.4) and (3.5),

$$[\gamma_{ns+1}^{(\mu)}]_{h_\kappa=0} = \begin{cases} 0, & 1 \le \mu < n \\ 1, & \mu = n \end{cases}, \tag{3.10}$$

and therefore, for $1 \le j \le \kappa - 1$, we have by (3.7)

$$\left[\frac{\partial a_{ns+1}^{(\kappa)}}{\partial h_j}\right]_{h^{(\kappa)}=0} = \left[\frac{\partial a_{ns+1}^{(\kappa-1)}}{\partial h_j}\right]_{h^{(\kappa-1)}=0} = \left[\frac{\partial a_{ns+1}^{(1)}}{\partial h_1}\right]_{h_1=0;\,\eta_1=\eta_j} \tag{3.11}$$

by the inductive hypothesis.

For Lemma 2 we use the fact[3] that $\zeta_s(h, \eta, z)$ is the solution of

$$s\frac{\partial\zeta_s(h,\eta,z)}{\partial h} + z\frac{1+\eta z^s}{1-\eta z^s}\frac{\partial\zeta_s(h,\eta,z)}{\partial z} = \zeta_s(h,\eta,z) \tag{3.12}$$

for which $\zeta_s(0, \eta, z) = z$. Substituting from (2.2) into (3.12) and equating like powers of z, we have

$$s\frac{\partial b_{ns+1}}{\partial h} = -nsb_{ns+1} - 2\sum_{\mu=1}^{n}[(n-\mu)s+1]b_{(n-\mu)s+1}\cdot\eta^\mu. \tag{3.13}$$

Setting $h = 0$ and using (2.4), we obtain Lemma 2.

[3] See Basilewitsch (1).

The theorem is now immediate. For by Lemma 1 it is sufficient to prove (3.1) with $j = k = 1$. We then have by (3.7), dropping the index 1,

$$\left[\frac{\partial a_{ns+1}}{\partial h}\right]_{h=0} = \sum_{\mu=0}^{n} a_{\mu s+1}^{(0)} \left[\frac{\partial}{\partial h}\left(\frac{\gamma_{ns+1}^{(\mu)}}{b_1}\right)\right]_{h=0}. \tag{3.14}$$

It is plain that we have to consider only those terms of $\gamma_{ns+1}^{(\mu)}$ which are of the form $h^m P(h)$, where m is zero or one; that is to say, terms of the form

$$b_1^l b_{ms+1}, \tag{3.15}$$

where necessarily, by (3.5),

$$l = \mu s, \qquad m = n - \mu. \tag{3.16}$$

Since, if $0 \leq \mu < n$, there are $(\mu s + 1)$ different ways of arranging $b_{(n-\mu)s+1}$ with μs b_1's, we see that the only terms which contribute to the right-hand side of (3.14) are

$$\begin{cases} (\mu s + 1) b_1^{\mu s} b_{(n-\mu)s+1}, & 0 \leq \mu < n \\ b_1^{ns+1}, & \mu = n \end{cases}. \tag{3.17}$$

Hence, substituting from (3.17) into (3.14), and recalling (2.3),

$$\begin{aligned} \left[\frac{\partial a_{ns+1}}{\partial h}\right]_{h=0} &= \sum_{\mu=0}^{n-1} (\mu s + 1) a_{\mu s+1}^{(0)} \left[\frac{\partial}{\partial h}\left(1 - \frac{h}{s}\right)^{\mu s-1} b_{(n-\mu)s+1}\right]_{h=0} - n a_{ns+1}^{(0)} \\ &= -\frac{2}{s} \sum_{\mu=0}^{n-1} (\mu s + 1) a_{\mu s+1}^{(0)} \eta^{n-\mu} - n a_{ns+1}^{(0)} \end{aligned} \tag{3.18}$$

by Lemma 2. This completes the proof of Theorem 1.

4. Writing $\eta_j = e^{i\theta}$, we obtain from Theorem 1 by elementary calculations:

THEOREM 2. *If $s = 1$ and $n > 1$,*

$$Re\left[\frac{\partial a_n^{(k)}}{\partial h_j}\right]_{h^{(k)}=0} < 0 \tag{4.1}$$

unless $\theta = \pi$; if $s = 2$ and $n > 1$,

$$Re\left[\frac{\partial a_{2n+1}^{(k)}}{\partial h_j}\right]_{h^{(k)}=0} > 0 \tag{4.2}$$

for $\dfrac{2\pi}{n+1} < |\theta| < \dfrac{2\pi}{n+1}\left(1 + \dfrac{1}{n}\right)$.

From (2.9) it is obvious that the left-hand sides of (4.1) and (4.2) must both vanish when $\theta = \pi$.

We begin with the case $s = 1$.[4] We suppose that $n > 1$, write $\eta_j = e^{i\theta} = \cos\theta + i \sin\theta$, and have then to show that

$$\begin{aligned} P_1(\theta) &= Re\left\{n(n-1) + 2\sum_{\mu=1}^{n-1} \mu^2 \eta_j^{n-\mu}\right\} \\ &= n(n-1) + 2\{(n-1)^2 \cos\theta \\ &\qquad + (n-2)^2 \cos 2\theta + \cdots + 1^2 \cos(n-1)\theta\} \end{aligned} \tag{4.3}$$

[4] I am indebted to Dr. R. Salem for suggesting this simple proof of (4.1).

is positive unless $\theta = \pi$. Multiplying both sides of (4.3) by $2 \sin^2 \frac{\theta}{2}$, we have

$$2P_1(\theta) \sin^2 \frac{\theta}{2} = n(1 + \cos \theta) - (1 + 2 \cos \theta + 2 \cos 2\theta + \cdots + 2 \cos (n-1)\theta + \cos n\theta)$$

$$= n(1 + \cos \theta) - \frac{\sin (2n+1) \frac{\theta}{2}}{\sin \frac{\theta}{2}} + \cos n\theta \tag{4.4}$$

$$= 2n \cos^2 \frac{\theta}{2} - \frac{\sin n\theta}{tg \frac{\theta}{2}}.$$

Finally, multiplying both sides of (4.4) by $\operatorname{tg} \frac{\theta}{2}$, we obtain

$$2P_1(\theta) \sin^2 \frac{\theta}{2} tg \frac{\theta}{2} = n \sin \theta - \sin n\theta. \tag{4.5}$$

Since $P_1(\theta)$ is an even function of θ, and since $P_1(0)$ is obviously positive, we may suppose that $0 < \theta < \pi$, in which case $\operatorname{tg} \frac{\theta}{2} > 0$, $\sin \theta > 0$, and $|\sin n\theta| < n \sin \theta$.

In the case $s = 2$, we write

$$P_2(\theta) = Re\left\{n + \sum_{\mu=0}^{n-1} (2\mu + 1)\eta_j^{n-\mu}\right\} \tag{4.6}$$

$$= n + (2n-1) \cos \theta + (2n-3) \cos 2\theta + \cdots + \cos n\theta,$$

and multiply both sides by $\sin^2 \frac{\theta}{2}$, obtaining

$$P_2(\theta) \cdot \sin^2 \frac{\theta}{2} = \cos \frac{\theta}{2} \sin \frac{n}{2} \theta \sin \frac{n+1}{2} \theta. \tag{4.7}$$

Writing $\theta = \frac{2\pi}{n+1} + \varphi$, we see that $P_2(\theta) < 0$ for $n > 1$ and

$$0 < \varphi < \frac{2\pi}{n(n+1)}. \tag{4.8}$$

Since $P_2(\theta)$ is even, this completes the proof of Theorem 2.

5. The results stated in §1 are now immediate. For, given a fixed $n > 1$, we have only to choose a value of $\eta_1 = e^{i\theta}$ for which (4.2) (with $j = k = 1$) is true, and then for all sufficiently small h_1 the absolute value of the $(2n+1)$—the coefficient of $f_1^{(1)}$ exceeds unity.

On the other hand, when $s = 1$, the situation is quite different. For

$$\left[\frac{\partial}{\partial h_j}|a_n^{(k)}|^2\right]_{h^{(k)}=0} = \left[\frac{\partial}{\partial h_j}\{a_n^{(k)}\bar{a}_n^{(k)}\}\right]_{h^{(k)}=0} = \left[a_n^{(k)}\frac{\partial \bar{a}_n^{(k)}}{\partial h_j} + \bar{a}_n^{(k)}\frac{\partial a_n^{(k)}}{\partial h_j}\right]_{h^{(k)}=0}$$

$$= 2Re\left[\bar{a}_n^{(k)}\frac{\partial a_n^{(k)}}{\partial h_j}\right]_{h^{(k)}=0} = 2n\cdot Re\left[\frac{\partial a_n^{(k)}}{\partial h_j}\right]_{h^{(k)}=0} \tag{5.1}$$

by (2.9). We suppose that $n > 1$, and that $\eta^{(k)} \neq -1$. Then by Theorem 2 there is a k-dimensional neighborhood $N = N(n, k, \eta^{(k)})$ of the point $h^{(k)} = 0$ in which

$$d\{|a_n^{(k)}|^2\} = 2n\cdot Re\sum_{j=1}^{k}\frac{\partial a_n^{(k)}}{\partial h_j}\,dhj < 0. \tag{5.2}$$

Now the function $f_1^{(0)}$ maps $|z| < 1$ on the plane slit along the negative real axis from $-\frac{1}{4}$ to ∞; and, interpreting the function $f_1^{(k)}$ geometrically, we see by (5.2) that *any* deformation of infinitesimal length of the slit of $f_1^{(0)}$ *decreases* $|a_n^{(0)}|$. We may state this result as follows:

THEOREM 3. *The functional* $|c_n| = |c_n(f_1)|$, $f_1 \in F_1$, *has a local maximum for* $f_1 = f_1^{(0)}$.

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ON SINGULAR MONOTONIC FUNCTIONS OF THE CANTOR TYPE

By R. Salem

This paper is devoted to the study of the Fourier-Stieltjes coefficients of continuous singular monotonic functions which are of the Cantor type, that is to say, which are constant in each interval contiguous to a perfect set of measure zero.[1] For the sake of brevity we shall thereafter call these functions: functions of the type C.

1. The first part of the paper deals with functions of the type C whose Fourier-Stieltjes coefficients tend to zero. It is well known (see Zygmund, [1], p. 294) that this question is closely related to the existence of sets of multiplicity for trigonometrical series.

The first result in this direction was obtained in 1916 by Menchoff [1], who proved the existence of such sets by constructing a function of the type C with coefficients tending to zero.

In a paper published in 1927, Nina Bary ([1], p. 113) raised the question to know if there was any limitation to the rapidity with which the coefficients a_n, b_n, of a trigonometric series converging to zero outside a set of multiplicity, could tend to zero (other than the limitation resulting from the trivial condition $\Sigma(a_n^2 + b_n)^2 = \infty$). The question is equivalent to the following one: what can be the smallest order of magnitude of the Fourier-Stieltjes coefficients of a function of the type C. Menchoff's result implied that his function had coefficients of order $(\log n)^{-\frac{1}{2}}$.

In 1936 Littlewood [1] proved the existence of functions of the type C with Fourier-Stieltjes coefficients of order $n^{-\epsilon}$ ($\epsilon > 0$).

Later, Wiener and Wintner [1], [2], and Schaeffer [1] have given examples of singular monotonic functions having Fourier-Stieltjes coefficients of order $n^{-\frac{1}{2}+\epsilon}$ ($\epsilon > 0$) (and even $n^{-\frac{1}{2}}r(n)$, $r(n)$ being any function increasing infinitely); but all these functions, although singular, are strictly increasing, hence not of the Cantor type, and thus do not give the solution of the problem of the order of the coefficients for a trigonometric series converging to zero outside a set of multiplicity.

In this paper following partly Wiener and Wintner's method, we construct a function of the type C whose Fourier-Stieltjes coefficients are of order $n^{-\frac{1}{2}+\epsilon}$, $\epsilon > 0$ being as small as we please.[2] Clearly, owing to the Fischer-Riesz theorem, ϵ cannot be negative.

The construction which we use allows us to state, in the second part of the paper, some new results in the question of sets of uniqueness and sets of multiplicity.

[1] The first function of this type seems to have been constructed by Lebesgue. The name of functions of the Cantor type is, after Wiener and Wintner [2], adopted here.

[2] And even of order $n^{-\frac{1}{2}}\Omega(n)$, $\Omega(n)$ increasing infinitely as slowly as we please. See the remark at the end of §9.

2. The third part of the paper deals with functions of the type C whose Fourier-Stieltjes coefficients do not tend to zero. It has been proved (see Salem, [1]) that there exist functions of the type C for which

$$\limsup_{n=\infty} \int_0^{2\pi} \cos nx \, dF = F(2\pi) - F(0)$$

(which is, of course, the largest possible upper limit). These functions are connected with the theory of absolute convergence of trigonometrical series. We shall give here the construction of a function of the type C such that the frequency $\frac{p}{n_p}$ of the integers n_p, for which the limit of $\int_0^{2\pi} \cos n_p x \, dF$ is $F(2\pi) - F(0)$, tends to zero as slowly as we please, which turns out to be the best possible result.

I

We shall first state a number of lemmas; secondly, we shall give the properties of a certain type of perfect sets and of the functions of the type C connected with them; we shall then prove our theorem.

3. *Lemmas. Lemma I. In order to find a function of the type C such that*

$$c_n = \int_0^{2\pi} e^{nix} \, dF(x) = 0(n^{-\frac{1}{2}+\epsilon})$$

it is sufficient to find a function $G(y)$ of the type C such that if $\gamma_n = \int_0^{2\pi} e^{niy} \, dG(y)$, *we have*

$$|\gamma_1| + |\gamma_2| + \cdots + |\gamma_n| = 0(n^2).$$

This lemma is due to Wiener and Wintner [1]; $F(x)$ is obtained by replacing, in $G(y)$, y by a function of x, the function $x(y)$ being a second-degree polynomial such that $x(0) = 0$, $x(2\pi) = 2\pi$, and that the derivative $\frac{dx}{dy}$ lies between two positive bounds, when $0 \leq x \leq 2\pi$. We can take, e.g., $x = \frac{1}{3\pi}(y^2 + \pi y)$.

Lemma II. Let $f(x)$ be a function of period 2π, continuous and of bounded variation, let $\omega(\delta)$ be its modulus of continuity, and let a_n, b_n, be its Fourier coefficients, with $\rho_n^2 = a_n^2 + b_n^2$; then

$$\frac{1}{n}(\rho_1^2 + 2^2 \rho_2^2 + \cdots + n^2 \rho_n^2) < C.V. \, \omega\left(\frac{1}{n}\right)$$

V being the total variation of $f(x)$ in $(0, 2\pi)$ and C an absolute constant.

This classical result is due to Wiener. (See Zygmund, [1], p. 221.)

Lemma III. Let $H(x)$ be a continuous function never decreasing in $(0, 2\pi)$

let $H(0) = 0, H(2\pi) = 1$ *and let* $\omega(\delta)$ *be its modulus of continuity; let* $h_n = \int_0^{2\pi} e^{nix}\, dH$. *Then*

$$|h_1|^2 + |h_2|^2 + \cdots + |h_n|^2 < Cn\omega\left(\frac{1}{n}\right)$$

C being an absolute constant.

This lemma is an immediate consequence of lemma II; it is sufficient to consider the function $f(x) = H(x) - \frac{x}{2\pi}$ which is continuous, of bounded variation, and such that $f(0) = f(2\pi) = 0$ and to observe that

$$\int_0^{2\pi} e^{nix}\left(H - \frac{x}{2\pi}\right)dx = \frac{i}{n}\int_0^{2\pi} e^{nix}\, dH.$$

Lemma IV. Considering that h_n *is defined for every value of* n *(non necessarily integer), we have also*

$$|h_{\frac{1}{2}}|^2 + |h_1|^2 + |h_{\frac{3}{2}}|^2 + \cdots + |h_{n-\frac{1}{2}}|^2 + |h_n|^2 < Cn\omega\left(\frac{1}{n}\right)$$

C being an absolute constant.

This extension of lemma III is trivial; it is sufficient to observe that

$$\int_0^{\pi} e^{niy}\left[H(2y) - \frac{y}{\pi}\right]dy = \frac{i}{n}\int_0^{2\pi} e^{\frac{1}{2}(nix)}\, dH(x) + 0\left(\frac{1}{n^2}\right)$$

and to apply the lemma II to the function of period 2π equal to $H(2y) - \frac{y}{\pi}$ in $(0, \pi)$ and to 0 in $(\pi, 2\pi)$. The extension to the case in which the indexes n of h_n increase by fractions equal, e.g., to $\frac{1}{3}$, is obvious.

4. *Construction of symmetrical perfect sets of order* d.

Let d be an integer ≥ 1 and let ξ be a positive number inferior to $\frac{1}{d+1}$.

Let us mark, on an interval A, B of length $B - A = L$, $d + 1$ non overlapping "white" intervals of length $L\xi$, such that the first interval has its left-hand end point in A, and the last interval has its right-hand end point in B, the d "black" intervals which separate the white intervals being of equal length $L\frac{1 - (d+1)\xi}{d}$. Such a dissection will be called a (d, ξ) dissection of the given interval.

Let us make now a (d, ξ_1) dissection of the interval $(0, 2\pi)$, let us remove the black intervals, and let us call $\eta_0^1, \eta_1^1, \cdots \eta_d^1$ the white intervals left. In a second step, we make a (d, ξ_2) dissection of each of the $d + 1$ intervals η_*^1; we remove the black intervals and we call $(\eta_*^1\eta_0^2), (\eta_*^1\eta_1^2) \cdots (\eta_*^1\eta_d^2)$ the white intervals left in η_*^1. Proceeding in the same way, after p operations, there are $(d+1)^p$ intervals left; each of them is called $(\eta_{\theta_1}^1\eta_{\theta_2}^2 \cdots \eta_{\theta_p}^p)$ where the θ_i take all the values $0, 1, \cdots d$. Each of these intervals is of length $2\pi\xi_1\xi_2 \cdots \xi_p$. It is clear

that proceeding in this way infinitely, and supposing always $0 < \xi_p < \frac{1}{d+1}$ we get a perfect set P nowhere dense, which is of measure zero if

$$\lim_{p=\infty} \xi_1 \xi_2 \cdots \xi_p (d+1)^p = 0$$

which we shall suppose henceforth.

It is seen immediately that the left-hand end points of the $(d+1)^p$ intervals left after the p^{th} operation have their abscissae given by the finite sum of p terms

$$2\pi\left[\theta_1 \frac{1-\xi_1}{d} + \theta_2 \frac{\xi_1(1-\xi_2)}{d} + \cdots + \theta_p \frac{\xi_1 \xi_2 \cdots \xi_{p-1}(1-\xi_p)}{d}\right]$$

where the θ_i take all values $0, 1, \cdots d$. Actually, this expression gives the left-hand end point of the interval called $(\eta^1_{\theta_1} \eta^2_{\theta_2} \cdots \eta^p_{\theta_p})$.

Also the abscissa of any point of the set P is given by the infinite series

$$(1)\quad x = 2\pi\left[\theta_1 \frac{1-\xi_1}{d} + \theta_2 \frac{\xi_1(1-\xi_2)}{d} + \cdots + \theta_p \frac{\xi_1 \cdots \xi_{p-1}(1-\xi_p)}{d} + \cdots\right]$$

where the θ_i can take all values $0, 1, \cdots d$. If the θ_i are all equal to d for $i \geq p+1$ we get the right-hand end points of the white intervals $(\eta^1_{\theta_1} \cdots \eta^p_{\theta_p})$.

5. *Construction of a function of the type C.*

Let $\lambda(1), \lambda(2) \cdots \lambda(d+1)$ be $d+1$ non negative numbers satisfying the two following conditions:

a) All the $\lambda(s)$ are $\leq \mu$, μ being a fixed number which is essentially inferior to 1 $(s = 1, 2, \cdots, d+1)$.

b) We have $\lambda(1) + \lambda(2) + \cdots + \lambda(d+1) = 1$. Moreover we define $\lambda(0) = 0$.

Our function $\Phi(x)$ will be the limit, for $p = \infty$, of the function $\Phi_p(x)$ defined in the following way: $\Phi_p(0) = 0$, $\Phi_p(2\pi) = 1$; $\Phi_p(x)$ is continuous, increases linearly by $\lambda(\theta_1+1)\lambda(\theta_2+1)\cdots\lambda(\theta_p+1)$ in the white interval $(\eta^1_{\theta_1} \eta^2_{\theta_2} \cdots \eta^p_{\theta_p})$ obtained at the p^{th} stage of the construction of the set, and is constant in every black interval. Since $|\Phi_p' - \Phi_{p+1}| \leq \mu^p$, $\Phi_p(x)$ tends uniformly to $\Phi(x)$ which is thus continuous, non decreasing in $(0, 2\pi)$, constant in every interval contiguous to P. Hence $\Phi(x)$ is of the type C. We have $\Phi(0) = 0$, $\Phi(2\pi) = 1$.

It is easily seen that when x belongs to P and is given by (1), we have

$$(2)\quad \begin{aligned}\Phi(x) = {} & [\lambda(0) + \lambda(1) + \cdots + \lambda(\theta_1)] + \lambda(\theta_1+1)[\lambda(0) + \cdots + \lambda(\theta_2)] \\ & + \lambda(\theta_1+1)\lambda(\theta_2+1)[\lambda(0) + \cdots + \lambda(\theta_3)] + \cdots\end{aligned}$$

the series being obviously convergent.

In the particular case in which all the $\lambda(s)$ are equal to $\frac{1}{d+1}$, we have

$$\Phi(x) = \frac{\theta_1}{d+1} + \frac{\theta_2}{(d+1)^2} + \cdots + \frac{\theta_p}{(d+1)^p} + \cdots$$

which is a classical function whose construction is due to Carleman ([1], p. 223), and earlier, in the case $d = 1$, to Hausdorff ([1], p. 173).

6. It is easy to give an estimation of the modulus of continuity of $\Phi(x)$.[3] The result is particularly simple in the case in which all the ξ_p are equal to a given $\xi < \frac{1}{d+1}$. Making this assumption, let x and $x' > x$ be two points of the interval $(0, 2\pi)$, and let us suppose first that they both belong to P. We have

$$x' - x = \frac{2\pi}{d}[(\theta'_{p+1} - \theta_{p+1})\xi^p(1-\xi) + (\theta'_{p+2} - \theta_{p+2})\xi^{p+1}(1-\xi) + \cdots]$$

if the p first θ_i are equal for x and x'. Now, if $x' > x$, $\theta'_{p+1} - \theta_{p+1} \geq 1$; on the other hand $|\theta'_{p+k} - \theta_{p+k}| \leq d$ hence

$$x' - x \geq \frac{2\pi}{d}\xi^p(1-\xi) - 2\pi[\xi^{p+1}(1-\xi) + \xi^{p+2}(1-\xi) + \cdots]$$

$$= \frac{2\pi}{d}\xi^p(1-\xi) - 2\pi\xi^{p+1} = \frac{2\pi}{d}\xi^p[1 - (d+1)\xi] = A\xi^p$$

$A > 0$ being independent of x and x'. Now, by (2)

$$\Phi(x') - \Phi(x) \leq \mu^p + \mu^{p+1} + \cdots = \frac{\mu^p}{1-\mu} = B\mu^p$$

B being independent of x and x'. We have, thus

$$p \geq \frac{-\log(x'-x)}{|\log \xi|} + \frac{\log A}{|\log \xi|}$$

hence

$$\Phi(x') - \Phi(x) \leq C\mu^{-\frac{\log(x'-x)}{|\log \xi|}} = C(x'-x)^{\frac{|\log \mu|}{|\log \xi|}} \tag{3}$$

C being a constant independent of x and x'. If x or x' or both, do not belong to P, let x_0 be the right-hand end point of the black interval containing x, and x'_0 the left hand end point of the black interval containing x'. As $x' - x > x'_0 - x_0$ and $\Phi(x') - \Phi(x) = \Phi(x'_0) - \Phi(x_0)$, the conclusion (3) subsists. Hence $\Phi(x)$ satisfies a Lipschitz condition of order $\frac{|\log \mu|}{|\log \xi|}$, verified inferior to 1, since necessarily $\mu \geq \frac{1}{d+1}$ and by hypothesis $\xi < \frac{1}{d+1}$.

7. Let us now calculate the Fourier-Stieltjes coefficients of Φ, that is to say the integral $\int_0^{2\pi} e^{nix}\, d\Phi(x)$. Putting $y = \Phi(x)$ we divide the interval $(0, 1)$

[3] The method is the one which has been used first by Hausdorff [1] in the determination of the "dimension" of a set.

of the y-axis by the points of subdivision which correspond to the black intervals removed after the p^{th} operation in the construction of our set. We have seen that $\Phi(x)$ increases by $\lambda(\theta_1+1)\lambda(\theta_2+1)\cdots\lambda(\theta_p+1)$ in the interval $(\eta^1_{\theta_1}\eta^2_{\theta_2}\cdots\eta^p_{\theta_p})$ whose left hand end point is

$$2\pi\left[\theta_1\frac{1-\xi_1}{d}+\theta_2\frac{\xi_1(1-\xi_2)}{d}+\cdots+\theta_p\frac{\xi_1\cdots\xi_{p-1}(1-\xi_p)}{d}\right]$$

Hence we have, for approximate expression of our integral

$$\sum\lambda(\theta_1+1)\lambda(\theta_2+1)\cdots\lambda(\theta_p+1)\exp\left\{\frac{2\pi ni}{d}[\theta_1(1-\xi_1)+\cdots+\theta_p\xi_1\cdots\xi_{p-1}(1-\xi)]\right\} \tag{4}$$

the summation being extended to the $(d+1)^p$ combinations of the values $0, 1, \cdots d$ of the θ_i. Putting

$$Q(\varphi)=\lambda(1)+\lambda(2)e^{i\varphi}+\lambda(3)e^{2i\varphi}+\cdots+\lambda(d+1)e^{di\varphi}$$

the expression (4) is equal to the product

$$\prod_{k=1}^{p}Q\left[\frac{2\pi n}{d}\xi_1\cdots\xi_{k-1}(1-\xi_k)\right]$$

and letting p increase infinitely, we have

$$\int_0^{2\pi}e^{nix}d\Phi(x)=\prod_{k=1}^{\infty}Q\left[\frac{2\pi n}{d}\xi_1\cdots\xi_{k-1}(1-\xi_k)\right].$$

In the particular case where all $\lambda(s)$ are equal to $\frac{1}{d+1}$ we have

$$Q(\varphi)=\frac{1}{d+1}\frac{e^{(d+1)i\varphi}-1}{e^{i\varphi}-1}=e^{\frac{1}{2}di\varphi}\frac{\sin(d+1)\frac{\varphi}{2}}{(d+1)\sin\frac{\varphi}{2}}$$

and if we denote by $\Phi_1(x)=\Phi_1(d,x)$ the particular function thus obtained, we have

$$\int_0^{2\pi}e^{nix}d\Phi_1(x)=e^{\pi ni}\prod_{k=1}^{\infty}\frac{\sin(d+1)\frac{\pi n}{d}\xi_1\cdots\xi_{k-1}(1-\xi_k)}{(d+1)\sin\frac{\pi n}{d}\xi_1\cdots\xi_{k-1}(1-\xi_k)} \tag{5}$$

which is the formula obtained by Carleman ([1], p. 225).

8. We shall need also the expression of the Fourier-Stieltjes coefficients of some other functions.

a) Let us take $d=2g$, g being an integer and let us take for $\lambda(s)$ the coefficient of z^{s-1} in the expansion of $\frac{1}{(g+1)^2}(1+z+z^2+\cdots+z^g)^2$, $(s=1,$

$2, \cdots, 2g + 1)$. Thus, $\lambda(1) + \lambda(2) + \cdots + \lambda(2g + 1) = 1$, as required. Here we have

$$Q(\varphi) = \left(\frac{1 + e^{i\varphi} + \cdots + e^{gi\varphi}}{g + 1}\right)^2 = \left[e^{\frac{gi\varphi}{2}} \frac{\sin (g + 1)\frac{\varphi}{2}}{(g + 1)\sin \frac{\varphi}{2}}\right]^2.$$

Hence, denoting the function by $\Phi_2(x) = \Phi_2(g, x)$,

$$(6) \qquad \int_0^{2\pi} e^{nix}\, d\Phi_2(x) = e^{\pi ni} \prod_{k=1}^{\infty} \left[\frac{\sin (g + 1)\frac{\pi n}{2g}\, \xi_1 \cdots \xi_{k-1}(1 - \xi_k)}{(g + 1)\sin \frac{\pi n}{2g}\, \xi_1 \cdots \xi_{k-1}(1 - \xi_k)}\right]^2$$

the function $\Phi_2(x)$ being of the type C if $\xi_p < \dfrac{1}{2g + 1}$.

b) In the same way, we can construct the function $\Phi_3(x) = \Phi_3(l, x)$ such that $d = 3l$ and $\lambda(s)$ is the coefficient of z^{s-1} in the expansion of

$$\frac{1}{(l + 1)^3}(1 + z + \cdots + z^l)^3$$

for $s = 1, 2, \cdots, 3l + 1$. We get

$$(7) \qquad \int_0^{2\pi} e^{nix}\, d\Phi_3(x) = e^{\pi ni} \prod_{k=1}^{\infty} \left[\frac{\sin (l + 1)\frac{\pi n}{3l}\, \xi_1 \cdots \xi_{k-1}(1 - \xi_k)}{(l + 1)\sin \frac{\pi n}{3l}\, \xi_1 \cdots \xi_{k-1}(1 - \xi_k)}\right]^3$$

the function $\Phi_3(x)$ being of the type C if $\xi_p < \dfrac{1}{3l + 1}$.

The generalization is obvious.

c) Another simple example is obtained in constructing, on the symmetrical perfect set of order d the function $\Psi(x) = \Psi(d, x)$ such that $\lambda(s)$ is the coefficient of z^{s-1} in the expansion of $\dfrac{1}{2^d}(1 + z)^d$. In this case

$$(8) \qquad \int_0^{2\pi} e^{nix}\, d\Psi(x) = e^{\pi ni} \prod_{k=1}^{\infty} \left[\cos \frac{\pi n}{d}\, \xi_1 \cdots \xi_{k-1}(1 - \xi_k)\right]^d$$

for $Q(\varphi) = \left(e^{\frac{i\varphi}{2}} \cos \dfrac{\varphi}{2}\right)^d$. The function $\Psi(x)$ is of the type C if $\xi_p < \dfrac{1}{d + 1}$.

9. *Proof of the theorem.* We can now give the proof of our theorem. Let g be an integer and let us construct the function of the type C

$$G(x) = \Phi_2(g, x)$$

on a set for which $d = 2g$ and all the ξ_p are equal to a given $\xi < \dfrac{1}{2g + 1}$. Let

$$\gamma_n = \int_0^{2\pi} e^{nix}\, dG(x).$$

We shall have, by (6)

$$\gamma_n = e^{\pi n i} \prod_{k=1}^{\infty} \left[\frac{\sin (g+1) \frac{\pi n}{2g} \xi^{k-1}(1-\xi)}{(g+1) \sin \frac{\pi n}{2g} \xi^{k-1}(1-\xi)} \right]^2. \tag{9}$$

Let us consider also the function

$$H(x) = \Phi_1(g, x)$$

defined in §7, constructed on a set for which $d = g$ and all the ξ_p are equal to *the same* ξ as above (which, being inferior to $\frac{1}{2g+1}$ is a fortiori inferior to $\frac{1}{g+1}$, as required). Let

$$h_n = \int_0^{2\pi} e^{nix}\, dH(x).$$

We shall have, by (5)

$$h_n = e^{\pi n i} \prod_{k=1}^{\infty} \frac{\sin (g+1) \frac{\pi n}{g} \xi^{k-1}(1-\xi)}{(g+1) \sin \frac{\pi n}{g} \xi^{k-1}(1-\xi)} \tag{10}$$

hence, by (9) and (10)

$$\gamma_n = \left[h_{\frac{n}{2}} \right]^2.$$

But we have seen that $H(x)$ satisfies a Lipschitz condition of order $\frac{\log (g+1)}{|\log \xi|}$, hence, by lemma IV,

$$\left| h_{\frac{1}{2}} \right|^2 + |h_1|^2 + \left| h_{\frac{3}{2}} \right|^2 + \cdots + \left| h_{\frac{n}{2}} \right|^2 = 0\left(n^{1-\frac{\log (g+1)}{|\log \xi|}}\right)$$

that is to say

$$|\gamma_1| + |\gamma_2| + \cdots + |\gamma_n| = 0\left(n^{1-\frac{\log (g+1)}{|\log \xi|}}\right).$$

Now we can take for ξ any number inferior to $\frac{1}{2g+1}$; taking $\xi = \frac{1}{2g+2}$, we observe that the expression

$$\epsilon = 1 - \frac{\log (g+1)}{\log (2g+2)} = \frac{\log 2}{\log (2g+2)}$$

is as near to zero as we please, since we can take g as large as we please. Hence

$$|\gamma_1| + |\gamma_2| + \cdots + |\gamma_n| = 0(n^{\epsilon})$$

and since $G(x)$ is of the type C, applying the transformation of Lemma I we get a function F of the type C such that

$$c_n = \int_0^{2\pi} e^{nix}\, dF = 0(n^{-\frac{1}{2}+\epsilon})$$

which proves the theorem.

Remark. It is possible to complicate more the sets and the functions constructed in the §4 and 5. First, we can change $\bar{d}$ from one step to another, the set being obtained by the dissections (d_1, ξ_1), $(d_2, \xi_2) \cdots (d_p, \xi_p) \cdots$ with $0 < \xi_p < \dfrac{1}{d_p + 1}$ and

$$\lim \xi_1 \xi_2 \cdots \xi_p (d_1 + 1) \cdots (d_p + 1) = 0.$$

Secondly, we can take a new set of values of λ, namely $\lambda_p(1)$, $\lambda_p(2)$, $\cdots$ $\lambda_p(d_p + 1)$, at every approximation Φ_p of the function Φ (notations of §5) the function Φ_p increasing by $\lambda_1(\theta_1 + 1)\lambda_2(\theta_2 + 1) \cdots \lambda_p(\theta_p + 1)$ in the interval $(\eta^1_{\theta_1}\eta^2_{\theta_2} \cdots \eta^p_{\theta_p})$. Here θ_p takes all values $0, 1, \cdots, d_p$.

The formulae are slightly more complicated with this generalization. But it is easy to see that by taking $\lambda_p(s) = \dfrac{1}{d_p + 1}$ for $s = 1, 2, \cdots d_p + 1$ and $\xi_p = \dfrac{1}{2d_p + 2}$, we get a function of the type C with a modulus of continuity $\omega(\delta)$ such that $n\omega\left(\dfrac{1}{n}\right)$ increases as slowly as we please. For in this case, with the same notations as in §6, we have

$$\frac{3A}{2^p} \frac{1}{(d_1 + 1) \cdots (d_p + 1)} \frac{\theta'_{p+1} - \theta_{p+1}}{d_{p+1}} > x' - x$$

$$> \frac{A}{2^p} \frac{1}{(d_1 + 1) \cdots (d_p + 1)} \frac{\theta'_{p+1} - \theta_{p+1}}{d_{p+1}};$$

$$\Phi(x') - \Phi(x) < \frac{B}{(d_1 + 1) \cdots (d_p + 1)} \frac{\theta'_{p+1} - \theta_{p+1}}{(d_{p+1} + 1)}$$

and it suffices to let d_p increase very rapidly. Thus, with the same method, we can obtain a function of the type C with coefficients of order $n^{-\frac{1}{2}}\Omega(n)$, $\Omega(n)$ increasing infinitely as slowly as we please.

II

10. The formulae proved in §8 will lead us to some other results. I am indebted to Prof. Zygmund for having drawn my attention to the problems which are thus solved here. Let us consider the function $\Psi(d, x)$ defined at

the end of §8, and constructed on a set for which all the ξ_p are equal to a given $\xi < \dfrac{1}{d+1}$. Let

$$C_n(d) = \int_0^{2\pi} e^{nix}\, d\Psi(x) = e^{\pi n i} \prod_{k=1}^{\infty} \left[\cos \frac{\pi n}{d}\, \xi^{k-1}(1-\xi) \right]^d$$

we have

$$C_n(d) = [C_{n/d}\,(1)]^d. \tag{11}$$

It is known (see Kershner [1], Nina Bary [2], p. 707) that if $\xi < \frac{1}{2}$ is rational and equal to the irreducible fraction p/q, $C_{n/d}\,(1)$ tends to zero for $n = \infty$ if $p \neq 1$ and does not tend to zero if $p = 1$.

The formula (11) proves that $C_n(d)$ behaves in the same way. Hence, as in the case $d = 1$, if $\xi = \dfrac{p}{q}$ $(p \neq 1)$ the symmetrical perfect set of order d and of constant ratio of dissection ξ, is a set of multiplicity. The only difference is that here we must have $\xi < \dfrac{1}{d+1}$.

11. In the case $d = 1$, the sets obtained by taking $\xi = \dfrac{1}{q}$ $(q > 2)$ are known (see Nina Bary, [2], p. 713) to be sets of the type H (see for the definition of such sets Zygmund [1], p. 268 and 292) and hence to be sets of uniqueness for trigonometrical series. It seems interesting to know if this is also true for $d > 1$, $\xi = \dfrac{1}{q}$ $(q > d+1)$.

We shall prove here that the sets for which $d > 1$, $\xi = \dfrac{1}{q}$, q integer $> d+1$, are sets of uniqueness (although not necessarily sets of the type H). In fact, let E be such a set. If x belongs to E, we have, by (1)

$$x = \frac{2\pi}{d}\,(q-1)\left[\frac{\theta_1}{q} + \frac{\theta_2}{q^2} + \cdots\right] \qquad (\theta_i = 0, 1, \cdots d). \tag{12}$$

Let E_1 be the set of the points

$$x_1 = 2\pi\left[\frac{\theta_1}{q} + \frac{\theta_2}{q^2} + \cdots\right] \qquad (\theta_i = 0, 1, \cdots d)$$

E_1 lies in the interval $\left(0,\ 2\pi\,\dfrac{d}{q-1}\right)$. Now, k being any integer > 0, we have

$$q^k x_1 \equiv 2\pi\left[\frac{\theta_{k+1}}{q} + \frac{\theta_{k+2}}{q^2} + \cdots\right] \pmod{2\pi}$$

that is to say

$$q^k x_1 \equiv y_1 \pmod{2\pi} \tag{13}$$

where y_1 belongs also to E_1. Now E_1 is, like E, a perfect set of measure zero constructed in $(0, 2\pi)$. Let Δ be the length of its greatest contiguous interval. The set of the points $q^k x_1 \pmod{2\pi}$ has a contiguous interval of length at least equal to Δ. This being true for every k, the set E_1 is of the type H (see Rajchman, [1], p. 404) hence it is a set of uniqueness.

Now the set E is derived from E_1 by a homothetic transformation $x = \frac{q-1}{d} x_1$; moreover, it lies entirely in $(0, 2\pi)$. Hence, by a theorem of Marcinkiewicz and Zygmund [1], it is a set of uniqueness (see also Nina Bary, [2], p. 714).

This leads to an interesting conclusion. The set constructed in §9 with $d = 2g$, $\xi = \frac{1}{2g+2}$ is a set of uniqueness. We have seen that (by application of lemma I) the mapping

$$z = \frac{x^2 + \pi x}{3\pi} \tag{14}$$

transforms this set in a set of multiplicity.[4] Hence we have the following result: *a very simple mapping of the type* (14) *can transform a set of uniqueness in a set of multiplicity.*

III

12. Let $F(x)$ be a function of the type C, $F(0) = 0$, $F(2\pi) = 1$ and $c_n = \int_0^{2\pi} e^{nix}\, dF$. What can be the frequency of integers n_p such that

$$\lim c_{n_p} = 1.$$

It is known (see Salem, [1]) that such a sequence of integers n_p exists for every $F(x)$ of the type C constructed on a perfect set of absolute convergence of trigonometrical series.

On the other hand, the application of the lemma III stated above and of Schwartz's inequality gives

$$\frac{1}{n}(|c_1| + |c_2| + \cdots + |c_n|) < C\sqrt{\omega\left(\frac{1}{n}\right)}$$

C being an absolute constant and $\omega(\delta)$ the modulus of continuity of F. Hence p/n_p tends to zero. Our purpose is to show that there exist functions F for which p/n_p tends to zero as slowly as we please.

Let us consider an ordinary symmetrical perfect set of measure zero (that is to say a perfect set of the type considered in the first part of this paper, in the simplest case $d = 1$) and let us construct on this set the function corresponding to $\lambda(1) = \lambda(2) = \frac{1}{2}$. We have, by (5)[5]

$$c_n = e^{\pi n i} \prod_{k=1}^{\infty} \cos \pi n\, \xi_1 \cdots \xi_{k-1}(1 - \xi_k)$$

the ξ_k being such that $0 < \xi_k < \frac{1}{2}$ and $\lim 2^p \xi_1 \xi_2 \cdots \xi_p = 0$.

[4] Assuming that $g > 1$.

[5] A direct proof of this formula is given in Salem [1], p. 325.

Let $s_1, s_2, \cdots s_k$ be an increasing sequence of integers; let $\xi_k = \frac{1}{2^{s_k}}$ and $r_k = s_1 + s_2 + \cdots + s_k$. We have

$$c_{2n} = \prod_{k=1}^{\infty} \cos 2\pi n\left[\frac{1}{2^{r_{k-1}}} - \frac{1}{2^{r_k}}\right].$$

Let us write n in the dyadic system

$$n = \sum_{h=0}^{m} a_h 2^h$$

and let us consider the integers n satisfying the following conditions:

A) $\varphi(m)$ being an integer increasing infinitely with m we must have

$$a_h = 0 \quad \text{for} \quad 0 \leq h \leq \varphi(m) - 1.$$

B) Moreover

$$a_h = 0 \quad \text{for} \quad r_k - k \leq h \leq r_k - 1 \qquad \text{(for every } k\text{)}.$$

C) all the other a_h's can take both values 0 or 1. Now, by (A), if $r_k \leq \varphi(m)$, $\cos 2\pi n\left[\frac{1}{2^{r_{k-1}}} - \frac{1}{2^{r_k}}\right] = 1$. If $r_k > \varphi(m)$ we have

$$\frac{n}{2^{r_k}} = \text{integer} + \frac{a_{r_k-1}}{2} + \frac{a_{r_k-2}}{2^2} + \cdots + \frac{a_{r_k-k}}{2^k} + \frac{a_{r_k-(k+1)}}{2^{k+1}} + \cdots$$

and thus, by (B)

$$\left|\frac{n}{2^{r_k}} - \text{integer}\right| \leq \frac{1}{2^{k+1}} + \frac{1}{2^{k+2}} + \cdots = \frac{1}{2^k}$$

and

$$\left|\frac{n}{2^{r_{k-1}}} - \frac{n}{2^{r_k}} - \text{integer}\right| \leq \frac{1}{2^{k-1}} + \frac{1}{2^k} = \frac{3}{2^k}$$

that is to say

$$\cos 2\pi n\left[\frac{1}{2^{r_{k-1}}} - \frac{1}{2^{r_k}}\right] > \cos \frac{6\pi}{2^k}$$

and

$$c_{2n} > \prod \cos \frac{6\pi}{2^k}$$

the product being extended to all k such that $r_k > \varphi(m)$. Now, if $n \to \infty$, $m \to \infty$, $\varphi(m) \to \infty$ and $k \to \infty$ and thus $c_{2n} \to 1$.

Let us now compute how many integers inferior to n exist satisfying the conditions (A) (B) and (C). If

(15) $$r_k \leq m < r_{k+1}$$

the number of such integers is superior to

$$2^{m+1-\varphi(m)-[1+2+\cdots+k+(k+1)]} > 2^{m+1-\varphi(m)-k^2}.$$

k being defined by (15) is a certain function of m, $k = \psi(m)$ increasing infinitely with m. The density of the integers of the above-described type is

$$d(n) > \frac{2^{m+1-\varphi(m)-k^2}}{2^{m+1}} = 2^{-\varphi(m)-\psi^2(m)}.$$

Now $n \geq 2^m$, hence $m \leq \dfrac{\log n}{\log 2}$, and thus

$$d(n) > 2^{-\varphi\left(\frac{\log n}{\log 2}\right)-\psi^2\left(\frac{\log n}{\log 2}\right)}.$$

$\epsilon(n)$ being given tending to zero for $n = \infty$ as slowly as we please, we can always choose the functions φ and ψ such that

$$d(n) > \epsilon(n)$$

which proves our statement.

It follows (see Salem [1]) that there exist trigonometric series

$$\Sigma\rho_p \cos(n_p x - \alpha_p) \qquad (\rho_p > 0,\ \Sigma\rho_p = \infty) \tag{16}$$

converging absolutely in a non-enumerable set, the integers n_p being such that p/n_p tends to zero as slowly as we please. This result is interesting because it is known (see Salem [1]) that if the sequence n_p is very regular (e.g. $n_p = p^m$, m integer) the series (16) cannot converge absolutely at more than one point. Here the sequence $\{n_p\}$ can be as dense as possible owing to its irregularity.

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THE NUMBER OF TWO-TERMINAL SERIES-PARALLEL NETWORKS

By John Riordan and C. E. Shannon

One of the first attempts to list all electrical networks meeting certain specified conditions was made in 1892 by P. A. MacMahon[1] who investigated combinations of resistances in series and in parallel, giving without proof a generating function from which the number of such combinations could be determined and a table of the numbers for combinations with 10 or less elements.[2]

The series-parallel combinations do not exhaust the possible networks since they exclude all bridge arrangements like the Wheatstone net,[3] but they are an important subclass because of their simplicity. When the number of elements is less than 5, all networks are series-parallel; at 5 there is one bridge-type network, the Wheatstone net; as the number of elements rises, bridge type networks increase faster than series-parallel until at 9 e.g. bridge-type are about 40% of the total. It appears from this (and it is known to be true) that for a large number of elements, series parallel networks are a relatively small part of the total; nevertheless the problem of enumerating all networks is so difficult that an extended study of the series-parallel networks is welcome on the principle that a little light is better than none.

Apart from this, the series-parallel networks are interesting in themselves in another setting, namely the design of switching circuits.[4] Here it becomes important to know how many elements are required to realize any switching function $f(x_1, \cdots, x_n)$ of n variables—that is, a number $N(n)$ such that every one of the 2^{2^n} different functions f can be realized with N elements and at least one with no less. An upper bound for the number of two terminal networks with B branches determines a lower bound for N since the number of different networks we can construct with N branches, taking account of different assignments of variables to the branches, can not be exceeded by 2^{2^n}; there must be enough networks to go around. This general fact is equally true if we limit the networks to the series-parallel type, and since switching networks are particularly easy to design in this form, the number of elements necessary for series-parallel realization of a function is of immediate interest.

These considerations have led us to work out a proof of MacMahon's generating function, which is given in full below; to develop recurrences and schemes of computation from this with which to extend MacMahon's table; to investigate

[1] "The Combination of Resistances," The Electrician, April 8, 1892; cf. also Cayley, Collected Works, III, **203**, pp. 242–6 for development of the generating function in another problem.

[2] It may be noted here that the number for 10 elements is given incorrectly as 4984; the correct value, 4624, is shown in Table I below.

[3] Complete enumerations of all possible circuits of n elements with n small classified in various ways are given by R. M. Foster, "The Geometrical Circuits of Electrical Networks," Trans. A. I. E. E., **51** (1932), pp. 309–317.

[4] C. E. Shannon, "A Symbolic Analysis of Relay and Switching Circuits," Trans. A. I. E. E., **57** (1938), pp. 713–723.

the behavior of the series-parallel numbers when the number of elements is large, and finally to make the application to switching functions mentioned above. These subjects are treated in separate sections.

For brevity in what follows we use the initials s.p. for series-parallel, e.s. for essentially series, and e.p. for essentially parallel.[5]

1. Derivation of Generating Function

For a single element, obviously only one network, the element itself, is possible. For 2, 3 and 4 elements, Fig. 1 shows all the s.p. networks obtainable divided into e.s. and e.p. classes for reasons which will appear.

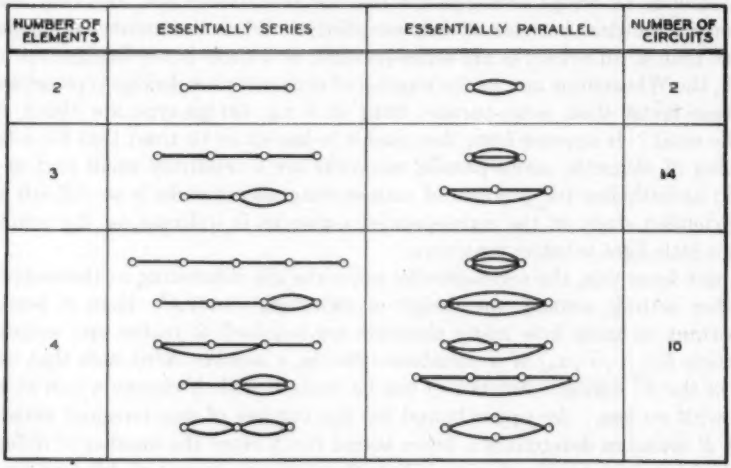

FIG. 1

It will be observed that no networks equivalent under series or parallel interchanges are listed; this is because for electrical purposes position in series or parallel is of no account.

[5] The concept of series-parallel connection is so intuitive that a formal definition seems unnecessary. However, since no definition seems to have been given in the literature, two equivalent definitions may be formulated as follows:

Definition I—A network N is series-parallel with respect to two terminals a and b if through each element of N there is at least one path from a to b not touching any junction twice, and no two of these paths pass through any element in opposite directions.

Definition II—A network is series-parallel if it is either a series or a parallel connection of two series-parallel networks. A single element is a series-parallel network.

Definition II is an inductive definition. Note that it serves to define equivalence under series-parallel interchanges directly; thus:

Two series-parallel networks are the same under series-parallel interchanges if they are series or parallel connections of the same two networks.

Note also the following:

A network is essentially series (essentially parallel) if it is the series (parallel) connection of two s.p. networks.

The classification exhibits a duality: e.s. and e.p. networks are equinumerous and in 1 − 1 correspondence. The rule of correspondence is that an e.s. network becomes e.p. when the words series and parallel in the description of the network are interchanged.

For enumerative purposes it is convenient to have a numerical representation of the networks. This may be done by using the sign + to denote elements in series, the dot or simple adjunction to denote elements in parallel, 1 to denote a circuit element, and abbreviating $1 + 1 + \cdots + 1$ (n elements in series) to n and $1 \cdot 1 \cdots 1$ (n elements in parallel) to 1^n; e.g. the symbol 21 represents a parallel connection of two elements in series and a single element.

Then the networks of Fig. 1 correspond in order to those in the following table:

n	*Essentially Series*	*Essentially Parallel*	*No. Cts.*
2	2	1^2	2
3	$3, 1^2 + 1$	$1^3, 21$	4
4	$4, 1^2 + 2, 21 + 1$ $1^3 + 1, 1^2 + 1^2$	$1^4, 21^2, (1^2 + 1)\,1$ $31, 22$	10

Fixing attention on the e.p. networks, it will be noticed that for $n = 2$ and 3, the representations are the partitions of n, excluding n itself. If the partition n itself is taken to mean the e.s. networks, then all s.p. networks are represented by the partitions of n, for $n < 4$. For $n = 4$ a non-partition e.p. representation $(1^2 + 1)\,1$ appears. But $1^2 + 1$ is one of the e.s. networks for $n = 3$. Hence all networks are included in the partition notation if each part of a partition is interpreted to mean the totality of the corresponding e.s. networks; e.g. the partition 31 is interpreted as the networks 31 and $(1^2 + 1)\,1$.

For enumerative purposes this means that each partition has a numerical coefficient attached to it determined by the number of e.s. networks corresponding to each of its component parts. If the number of e.s. networks for p elements is denoted by a_p, the coefficient for a partition $(pqr \cdots)$ where no parts are repeated is $a_p\, a_q\, a_r \cdots$ with $a_1 = a_2 = 1$, since each of the combinations corresponding to a given part may be put in parallel with those corresponding to the remaining parts. The coefficient for a repeated part, say p^π, p repeated π times, is the number of combinations π at a time of a_p things with unrestricted repetition, which is the binomial coefficient:[6]

$$\binom{a_p + \pi - 1}{\pi}$$

Hence the total number of s.p. networks s_n for n elements may be written as:

$$s_n = 2a_n = \sum \binom{a_p + \pi_1 - 1}{\pi_1} \binom{a_q + \pi_2 - 1}{\pi_2} \cdots \qquad (1)$$

[6] Netto, *Lehrbuch der Combinatorik*, Leipzig, 1901, p. 21 or Chrystal, *Algebra* II, London, 1926, p. 11.

where the sum is over all integral non-negative $p, q \cdots, \pi_1, \pi_2 \cdots$ such that

$$p\pi_1 + q\pi_2 + r\pi_3 + \cdots = n$$

and $a_1 = a_2 = 1$. That is, the sum is over all partitions of n.

Thus for $n = 5$ the partitions are:

$$5, 41, 32, 31^2, 2^21, 21^3, 1^5;$$

and

$$s_5 = a_5 + a_4 + a_3 + a_3 + 1 + 1 + 1,$$

or since $s_n = 2a_n$

$$s_5 = s_4 + 2s_3 + 6 = 24.$$

Similarly:

$$s_6 = s_5 + 2s_4 + 2\binom{a_3 + 1}{2} + 2s_3 + 8 = 66.$$

The generating function[7] given by MacMahon, namely:

$$\prod_1^\infty (1 - x^i)^{-a_i} = 1 + \sum_1^\infty s_n x^n \tag{2}$$

where $\prod$ signifies a product, may be derived from (1) by an argument not essentially different from that used for the Euler generating-function[8] for the partitions of n, which is

$$\prod_1^\infty (1 - x^i)^{-1} = 1 + \sum_1^\infty p_n x^n$$

2. Numerical Calculation

Direct computation from the generating identity (2) or its equivalent, equation (1), becomes cumbersome for relatively small values of n, since the number of terms is equal to the number of partitions. Moreover, the computation is serial, each number depending on its predecessors, involving cumulation of errors; hence independent schemes of computation are desirable.

The three schemes used in computing the series-parallel numbers shown in Table I[9] follow closely schemes for computing the number of partitions, namely those due respectively to Euler and Gupta, and that implicit in the recurrence formula.

[7] It should be observed that this is not a generating function in the sense that the coefficients of the power series are completely determined by expansion, but rather a generating identity determining coefficients by equating terms of like powers.

[8] Cf., for example, Hardy and Wright "An Introduction to the Theory of Numbers," Oxford, 1938, p. 272.

[9] We are indebted to our associate Miss J. D. Goeltz for the actual computation.

TABLE I

Series-Parallel and Associated Numbers

n	s_n	σ_n
1	1	1
2	2	1
3	4	1
4	10	3
5	24	5
6	66	17
7	180	41
8	522	127
9	1,532	365
10	4,624	1,119
11	14,136	3,413
12	43,930	10,685
13	137,908	33,561
14	437,502	106,827
15	1,399,068	342,129
16	4,507,352	1,104,347
17	14,611,576	3,584,649
18	47,633,486	11,701,369
19	156,047,204	38,374,065
20	513,477,502	126,395,259
21	1,696,305,720	
22	5,623,993,944	
23	18,706,733,128	
24	62,408,176,762	
25	208,769,240,140	
26	700,129,713,630	
27	2,353,386,723,912	
28	7,927,504,004,640	
29	26,757,247,573,360	
30	90,479,177,302,242	

The first depends essentially on the computation of an allied set of numbers $s_n(k)$ defined by:

$$\prod_1^k (1 - x^i)^{-a_i} = 1 + \sum_{n=1}^{\infty} s_n(k)x^n, \tag{3}$$

with $s_n = s_n(N), N \geq n$.

A recurrence formula for these numbers follows directly from the definition and reads as follows:

$$s_n(k) = \sum_{i=0}^{a_k} \binom{a_k + i - 1}{i} s_{n-ik}(k-1), \tag{4}$$

with q the integral part of n/k and $s_0(k-1) = s_0(k) = 1$. Clearly $s_n(1) = 1$, $s_n(2) = 1 + [\frac{1}{2}n]$, where the brackets indicate "integral part of."

Note that $s_n(k)$ enumerates the number of e.p. (or e.s.) networks with n elements that can be formed from parts no one of which contains more than k elements; e.g., the e.p. networks enumerated by $s_4(2)$ are 2^2, 21^2, and 1^4. This remark, coupled with the interpretation of the binomial coefficients given in Section 1, gives a ready network interpretation of the recurrence (4).

Although, as indicated, the numbers $s_n(k)$ may be used directly for computation of s_n, they are more efficiently used in the following formula:

$$s_n = s_{n-1} + s_{n-2}s_2 + \cdots + s_{n-m-1}s_{m+1} + 2s_n(m) \tag{5}$$

where $m = [\frac{1}{2}n]$.

The network interpretation of this is seen more readily in the equivalent form:

$$s_n = a_n + a_{n-1}a_1 + a_{n-2}s_2 + \cdots + a_{n-m-1}s_{m+1} + s_n(m) \tag{5.1}$$

Thus the total number of networks with n elements is made up of e.s. networks with n elements enumerated by a_n, plus e.p. networks formed by combining all e.s. networks of $n - i$ elements with all networks of i elements, $i = 1$ to the smaller of $m + 1$ and $n - m - 1$, plus finally the networks enumerated by $s_n(m)$ as described above.

This is essentially all that is used in what may be called the Euler computation.

The Gupta computation rests upon division of partitions into classes according to size of the lowest part; e.g. if the partitions of n with lowest part k are designated by $p_{n,k}$, then the classes for $n = 4$ are:

$$p_{4,1} = (31, 21^2, 1^4)$$

$$p_{4,2} = (2^2)$$

$$p_{4,3} = \text{None}$$

$$p_{4,4} = (4)$$

Recurrence formulae for the corresponding network classes $s_{n,k}$ are derived by appropriate modification of a procedure given by Gupta; thus e.g. if a unit is deleted from each of the partitions in $p_{n,1}$, the result is exactly p_{n-1}, hence:

$$s_{n,1} = s_{n-1}.$$

Similarly:

$$s_{n,2} = a_2[s_{n-2,2} + s_{n-3,3} + \cdots s_{n-2,n-2}]$$

$$= s_{n-2} - s_{n-2,1} = s_{n-2} - s_{n-3}.$$

In general:

$$s_{n,k} = \sum_{i=1}^{q} \binom{a_k + i - 1}{i} A_{n-ik,k}, \tag{6}$$

with

$$q = [n/k]$$

$$A_{0,k} = 1,$$

$$A_{r,k} = 0, \qquad r = 1, 2 \cdots k,$$

$$A_{r,k} = s_r - s_{r,1} - \cdots - s_{r,k}, \qquad r > k.$$

Another form of (6), derived by iteration and simpler than (6) for small values of k and large values of n, is as follows:

$$s_{n,k} = \sum_{i=1}^{q} \binom{a_k}{i} A_{n-ik,k-1}. \tag{6.1}$$

It should be noted that vacuous terms appear in the sum if $q > a_k$.

The third scheme of computation consists in determining a third set of numbers, σ_n, defined by:

$$\prod_1^\infty (1 - x^i)^{a_i} = 1 - \sum_1^\infty \sigma_n x^n \tag{7}$$

Coupling this definition with the MacMahon generating identity, equation (2), it follows that:

$$s_n = \sum_{i=1}^{n} \sigma_i s_{n-i}, \tag{8}$$

with s_0 taken by convention as unity.

The recurrence formula for these numbers is as follows:

$$\sigma_n = a_n - \sum_{i=1}^{n-m-1} \sigma_i a_{n-i} + \sigma_n(m) \tag{9}$$

where, as above $m = [\frac{1}{2}n]$ and $\sigma_n(k)$ is defined in a manner similar to $s_n(k)$. Note that $\sigma_1 = \sigma_2 = \sigma_3 = 1$. These numbers are included in Table I ($n < 20$).

3. Asymptotic Behavior

The behavior of s_n for large n is ideally specified by an exact formula or, failing that, an asymptotic formula. It is a remarkable fact that the asymptotic formula for the partition function is an "exact" formula, that is, can be used to calculate values for large n with perfect accuracy. We have not been able to find either for s_n; we give instead comparison functions bounding it from above and below.

It is apparent, first of all, that $s_n \geq p_n$ for all values of n. This is very poor. Somewhat better is

$$s_n \geq \pi_n \tag{10}$$

where $\pi_n = 2^{n-1}$ is the number of compositions of n, that is, partitions of n in which the order of occurrence of the parts is essential.

This is proved as follows. From equation (5), $s_n > q_n$ if

$$q_n \leq s_n \qquad n \leq 4$$

$$q_n = q_{n-1} + s_2 q_{n-2}, \qquad n > 4$$

The solution of the last, taking $q_3 = 4$, $q_4 = 8$, is:

$$q_n = 2^{n-1} = \pi_n$$

More terms of equation (5) push the lower bound up but at a relatively slow rate and the analysis increases rapidly in difficulty; the best we have been able to show in this way is:

$$s_n \geq A3^n \tag{11}$$

with A a fixed constant.

An alternative, more intuitive, way, however, is much better. First, notice that the networks for n elements are certainly greater in number than those obtained by connecting a single element in series or in parallel with the networks for $n - 1$ elements, a doubling operation; hence, $s_n \geq \pi_n$ where

$$\pi_n = 2\pi_{n-1} = 2^2\pi_{n-2} = 2^{n-1}\pi_1 = 2^{n-1},$$

which is the result reached above.

The networks of n elements with a single element in series or in parallel are exactly those enumerated by $s_{n,1}$ in the Gupta classification. Hence the approximation may be bettered by considering more terms in the expansion:

$$s_n = 2\sum_{i=1}^{m} s_{n,i}, \qquad m = [\tfrac{1}{2}n].$$

The term $s_{n,i}$ enumerates the e.s. networks in which the smallest e.p. part has exactly i elements. If this part is removed from each of these networks, the networks left are certainly not less than the e.s. networks with $n - i$ elements if $i < m$; that is

$$s_{n,i} \geq a_i a_{n-i} \qquad i < m$$

For n even, say $2m$;

$$s_{2m,m} = \binom{a_m + 1}{2} = \tfrac{1}{2}(a_m^2 + a_m) = \tfrac{1}{2}(s_m^2 + 2s_m);$$

for n odd;

$$s_{2m+1,\,m} = a_{m+1}a_m = \tfrac{1}{4}s_{m+1}s_m .$$

Hence:

$$\begin{aligned} s_{2m} &\geq 2\,s_{n-1} + \frac{1}{2}\sum_{i}^{m-1} s_i s_{n-i} + \frac{1}{4}(s_m^2 + 2s_m) \\ s_{2m+1} &\geq 2\,s_{n-1} + \frac{1}{2}\sum_{i}^{m} s_i s_{n-i} \end{aligned} \tag{12}$$

Then, in general, $s_n > r_n$ if $r_1 = 1, r_2 = 2$ and

$$r_n = \frac{3}{2} r_{n-1} + \frac{1}{4} \sum_1^{n-1} r_i r_{n-i}, \qquad n > 2 \tag{13}$$

Writing the generating function for the r_n as:

$$R(x) = \sum_1^{\infty} r_n x^n,$$

the recurrence (13) together with the initial conditions entail:

$$[R(x)]^2 - (4 - 6x)R(x) + 4x - x^2 = 0;$$

that is:

$$R(x) = 2 - 3x - 2\sqrt{1 - 4x + 2x^2} \tag{14}$$

The asymptotic behavior of the r_n may be determined from $R(x)$ by the method of Darboux[10] with the following result:

$$r_n \sim A\lambda^n n^{-3/2} \tag{15}$$

with A a fixed constant and $\lambda = 2 + \sqrt{2} = 3.414 \cdots$.

An upper bound follows by the same process on remarking that:

$$s_{n,i} \leq a_i s_{n-i}.$$

Hence, $s_n \leq t_n$ if $t_1 = 1, t_2 = 2$ and

$$t_n = t_{n-1} + \frac{1}{2} \sum_1^{n-1} t_i t_{n-i} \tag{16}$$

By the procedure followed above:

$$T(x) = \sum_{n=0}^{\infty} t_n x^n = 1 - x - \sqrt{1 - 4x} \tag{17}$$

and

$$t_n = \frac{4(2n-3)!}{n!(n-2)!} \quad n > 1$$

$$\sim \frac{2}{\sqrt{\pi}} 4^{n-1} n^{-3/2}$$

A comparison of r_n, s_n and t_n for $n \leq 10$, taking for convenience the integral part only of r_n (denoted by $[r_n]$) is as follows:

n	1	2	3	4	5	6	7	8	9	10
$[r_n]$	1	2	4	9	22	57	154	429	1225	3565
s_n	1	2	4	10	24	66	180	522	1532	4624
t_n	1	2	4	10	28	84	264	858	2860	9724

[10] Hilbert-Courant: Methoden der Mathematischen Physik I, pp. 460–2 (Springer, 1931, 2nd ed.).

Note that the lower bound is closer to the true value for the larger values of n.

A third, semi-empirical, bound is worth noting. From table I note that for $n > 2$, $4\sigma_n$ is approximately equal to s_n. Taking this as an equality and using equation (8) and the known values $\sigma_1 = \sigma_2 = 1$ the equation for the generating function $U(x)$ of the approximation u_n turns out to be:

$$U(x) = \tfrac{1}{2}[5 - 3x - 2x^2 - \sqrt{9 - 30x - 11x^2 + 12x^3 + 4x^4}] \qquad (18)$$

A comparison of s_n and the integral part of u_n is shown in Table II for $n \leq 20$. The approximation is remarkably close; the worst percentage difference is 10% for $n = 4$, but from $n = 7$ to 20 the agreement is within 3%.

TABLE II

Approximation to Series-Parallel Numbers

n	$[u_n]$	s_n
1	1	1
2	2	2
3	4	4
4	9	10
5	23	24
6	63	66
7	177	180
8	514	522
9	1,527	1,532
10	4,625	4,624
11	14,230	14,136
12	44,357	43,930
13	139,779	137,908
14	444,558	437,502
15	1,425,151	1,399,068
16	4,600,339	4,507,352
17	14,939,849	14,611,576
18	48,778,197	47,633,486
19	160,019,885	156,047,204
20	527,200,711	513,477,502

The asymptotic behavior of u_n is found to be:

$$u_n \sim A\lambda^n n^{-3/2}$$

with A about 3/7, λ about 3.56.

4. Series-Parallel Realization of Switching Functions

As an application of these results it will be shown now that almost all switching functions of n variables require at least

$$(1 - \epsilon)\frac{2^n}{\log_2 n} \qquad \epsilon > 0$$

switching elements (make or break contacts) for their realization in an s.p. network.

The number of functions that can be realized with h elements is certainly less than the number of s.p. networks s_h multiplied by the number of different ways that the elements in each network may be labeled. This latter number is $(2n)^h$ since each element has a choice of $2n$ labels corresponding to each variable and its negative. Hence, not more than

$$(2n)^h s_h \leq (2n)^h 4^h = (8n)^h$$

different functions can be realized with h elements. If

$$h = \frac{2^n}{\log_2 n}(1 - \epsilon) \qquad \epsilon > 0$$

the fraction of all 2^{2^n} functions of n variables that can be realized is less than.

$$\frac{(8n)^{\frac{2^n}{\log_2 n}(1-\epsilon)}}{2^{2^n}} = \frac{2^{3(1-\epsilon)2^n \log_n 2+(1-\epsilon)2^n}}{2^{2^n}}$$

$$< 2^{3.2^n \log_n 2-\epsilon \cdot 2^n}$$

and since this approaches zero as $n \to \infty$ for any positive ϵ, the result is proved

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ON THE FOURIER ANALYSIS OF ORBITS IN THE EQUATORIAL PLANE OF A MAGNETIC DIPOLE[1]

By Jaime Lifshitz[2]

1. Introduction

According to the theory of primary cosmic rays developed by G. Lemaitre and M. S. Vallarta,[3] such rays are made up of electrically charged particles moving at high speed. To find an orbit of these particles, it is necessary to study the motion of an electrically charged particle in the magnetic field of the earth which, as a first approximation, is that of a dipole.

The equations of motion of a particle in its own meridian plane are:[4]

$$\left(\frac{dx}{d\sigma}\right)^2 + \left(\frac{d\lambda}{d\sigma}\right)^2 = ae^{2x} - (e^{-x}\cos\lambda - \sec\lambda)^2 = P, \tag{1}$$

$$\frac{d^2x}{d\sigma^2} = ae^{2x} - e^{-x} + e^{-2x}\cos^2\lambda = X = \frac{1}{2}\frac{\partial P}{\partial x}, \tag{2}$$

$$\frac{d^2\lambda}{d\sigma^2} = e^{-2x}\sin\lambda\cos\lambda - \frac{\sin\lambda}{\cos^3\lambda} = \Lambda = \frac{1}{2}\frac{\partial P}{\partial \lambda}; \tag{3}$$

where σ is a monotonic function of time, x is a function of the radius ρ, λ is the latitude and $a = 1/16\gamma_1^4$ is the parameter appearing on integration of the differential equation governing the rotation of the meridian plane about the polar axis. This motion has no importance for our problem on account of the symmetry of the magnetic field with respect to its polar axis.

For values of the parameter γ_1 within the interval $0.78856 < \gamma_1 < 1$ there exist for each value of this parameter two periodic orbits symmetric with respect to $\lambda = 0$, discovered by Störmer,[5] which have been called principal periodic orbits. One of the families of such orbits, the so-called external, plays an important role in the theory of cosmic rays because their asymptotic orbits are those which separate trajectories coming from infinity from those which proceed for the vicinity of the dipole. Upon studying the symmetrical orbits for values of $\gamma_1 > 1$, which may be considered as a continuation of one of the families discovered by Störmer, the so-called internal orbits, it was found that, for the

[1] Based on a thesis presented in partial fulfilment for the requirements of the degree of Master of Mathematical Sciences in the Faculty of Sciences of the National University of Mexico.

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[3] G. Lemaitre and M. S. Vallarta, "On Compton's Latitude Effect of Cosmic Radiation," Phys. Rev. **43**, 87-91 (1933). This is the original paper with which Lemaitre and Vallarta started their research on cosmic radiation. For a complete bibliography of the papers of Lemaitre and Vallarta, see Alfredo Baños Jr., Journal of the Franklin Institute, **227**, No. 5 (1939), p. 644. An excellent summary by M. S. Vallarta was published in the University of Toronto Studies, Applied Mathematics Series No. 3 (1938).

[4] Alfredo Baños, Jr., Journal of the Franklin Institute, **227**, 643 (1939).

[5] Carl Störmer, Zeits. f. Astrophys. **1**, 237-274 (1930).

value of $\gamma_1 = 1.31359$, such orbits tend towards the equatorial orbits,[6] which in the meridian plane are all periodic for $\gamma_1 > 1$, and are represented as a segment of the axis $\lambda = 0$.

The equations of motion (1), (2) and (3) in general are not integrable. The special case of the equatorial orbits is an exception, for in this case the solution may be expressed in terms of elliptic functions.

In order to be able to study more in detail the behavior of orbits in the vicinity of the equatorial plane, an important factor is to investigate the stability of such orbits and also of the equatorial orbits, which quantitatively is given by the characteristic exponent. The characteristic exponent can be found from the variational equations which, for the equatorial orbits, are of the form:[7]

$$\frac{d^2\psi_1}{d\sigma^2} = (2ae^{2\psi} - 2e^{-2\psi} + e^{-\psi})\psi_1, \tag{4}$$

$$\frac{d^2\eta_1}{d\sigma^2} = (e^{-2\psi} - 1)\eta_1; \tag{5}$$

where ψ_1, η_1 are the perturbations of x and λ, and $x = \psi(\sigma)$ is the periodic solution of the case in question.

The motion in the equatorial plane is given by the equation:

$$e^{-x} = \frac{\alpha + \delta k \operatorname{sn}^2\left(\frac{2K}{\pi}\omega\sigma\right)}{1 + k \operatorname{sn}^2\left(\frac{2K}{\pi}\omega\sigma\right)}, \tag{6}$$

where

$$2\alpha = 1 + \sqrt{1 + 1/\gamma_1^2},$$

$$2\delta = 1 - \sqrt{1 + 1/\gamma_1^2},$$

$$k = \frac{\sqrt{1 + 1/\gamma_1^2} - \sqrt{1 - 1/\gamma_1^2}}{\sqrt{1 + 1/\gamma_1^2} + \sqrt{1 - 1/\gamma_1^2}} \qquad \text{modulus of sn}$$

K = one-fourth of a period of sn,

$$\omega = \frac{\alpha - \delta}{1 + k} \cdot \frac{\pi}{4K}.$$

In order to find the characteristic exponents, it is convenient to expand the coefficients of the variational equations in Fourier series. These equations then take the classical form of Hill's equations.[8] The Fourier series for the functions

[6] Unpublished work of the author carried out under the direction of Dr. Alfredo Baños, Jr., Director of the Institute of Physics.

[7] Odon Godart, Ann. de la Soc. Sci. de Bruxelles, **58**, 30 (1938).

[8] Whittaker and Watson, "A Course of Modern Analysis," Cambridge University Press, 4th ed., 1927, par. 19.41.

e^{2x}, e^{-x}, e^{-2x}, and x are given in the sequel. The first three are used to cast the variational equations in their canonical form, and the last one to verify the results through equation (2).

2. Expansion in Fourier Series

The functions e^{-x}, e^{-2x} and e^{2x} in terms of σ are easily obtained by means of equation (6). For real values of σ the three functions satisfy the Dirichlet conditions.[9] The variable $x = \log(e^x)$ also satisfies these conditions for real values of σ. The coefficients of the Fourier series can therefore be found by integration.

The double periodicity of these four functions leads to the required integrals through the residual theorem, which brings down the actual integration to the study of these functions in the vicinity of their singular points in the complex

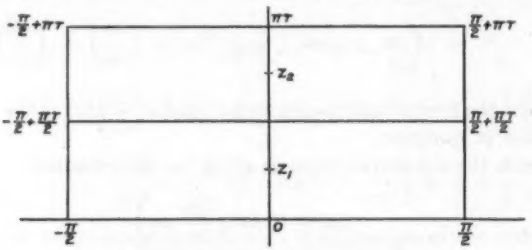

FIG. 1. Periodicity cell in z-plane

plane of the variable σ. The four functions and all their derivatives are continuous for real values of σ. This is the method used in Whittaker and Watson's "A Course of Modern Analysis," paragraph 22.6 for the expansion of the Jacobi functions in Fourier series.

2a. Calculation of the Fourier coefficients $a_{2n}(n \neq 0)$ for the function e^{-x}

From equation (6) we find the relation:

$$e^{-x} = f(\omega\sigma) = f(z) = \frac{\alpha + \delta k \operatorname{sn}^2\left(\frac{2Kz}{\pi}\right)}{1 + k \operatorname{sn}^2\left(\frac{2Kz}{\pi}\right)}.$$

Using the notation of Whittaker and Watson's chapter XXI:

$$i\frac{K'}{K} = \tau, \qquad q = e^{-\pi\frac{K'}{K}} = e^{\pi i\tau},$$

it is noticed that the function $f(z)$ has the periods π and $\pi\tau$; that is to say: $f(z + \pi) = f(z)$, $f(z + \pi\tau) = f(z)$; for this reason it is sufficient to study the function within the cell indicated in Figure 1.

[9] Whittaker and Watson, loc. cit., par. 9.1.

Since $sn^2\left(\frac{2Kz}{\pi}\right)$ is a meromorphic function,[10] the only singular points of $f(z)$ arise from the zeros of the denominator or the poles of the numerator. Since for the points where the numerator has a pole the denominator also has another pole of the same order, the second possibility is discarded. The only poles of $f(z)$ are then the roots of the equation

$$1 + k\,\mathrm{sn}^2\left(\frac{2Kz}{\pi}\right) = 0.$$

Within the cell the function has two simple poles:[11]

$$z_1 = \frac{\omega\tau}{4}, \quad z_2 = \frac{3\pi\tau}{4}.$$

Since z_1 is a simple pole, we have for its residual:[12]

$$R_1 = \lim_{z\to z_1}(z - z_1)f(z) = \lim_{z\to z_1}\left[\frac{\alpha + \delta k\,\mathrm{sn}^2\left(\frac{2Kz}{\pi}\right)}{1 + k\,\mathrm{sn}^2\left(\frac{2Kz}{\pi}\right)}\Big/(z - z_1)\right]$$

$$= \frac{\alpha + \delta k\,\mathrm{sn}^2\left(\frac{2Kz_1}{\pi}\right)}{\frac{d}{dz}\left(1 + k\,\mathrm{sn}^2\left(\frac{2Kz}{\pi}\right)\right)\Big|_{z=z_1}} = \frac{\alpha + \delta k\,\mathrm{sn}^2\left(\frac{2Kz_1}{\pi}\right)}{2k\frac{2K}{\pi}\,\mathrm{sn}\frac{2Kz_1}{\pi}\,\mathrm{cn}\frac{2Kz_1}{\pi}\,\mathrm{dn}\frac{2Kz_1}{\pi}}$$

$$= \frac{(\alpha - \delta)\pi}{2k\cdot 2Ki\frac{1+k}{k}} = -i\frac{\alpha - \delta}{1+k}\cdot\frac{\pi}{4K} = -i\omega.$$

The value of the residual for the other pole is easily inferred from the theorem that the sum of the residuals of an elliptic function within a cell vanishes, and therefore:

$$R_2 = -R_1 = i\omega.$$

Since π is the period of the function $f(z)$, the odd harmonics of its expansion in a Fourier series vanish, and the even ones are given by the formulas:

$$a_{2n} = \frac{2}{\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} f(z)\cos 2nz\,dz.$$

By virtue of Euler's formulas, the preceding expression is the same as the real part of the integral:

$$\frac{2}{\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} f(z)e^{2niz}\,dz.$$

[10] E. Goursat, "Cours d'Analyse Mathématique," Paris: Gauthier-Villars, 5th ed., 1929. Vol. 2, par. 292.

[11] Whittaker and Watson, loc. cit., par. 22.41, ex. 2.

[12] E. Goursat, loc. cit., par. 294.

Integrating along the contour C: $\left(-\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}+\pi\tau, -\frac{\pi}{2}+\pi\tau, -\frac{\pi}{2}\right)$, we shall have

$$\int_C f(z)e^{2niz}\,dz = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} + \int_{\frac{1}{2}\pi}^{\frac{1}{2}\pi+\pi\tau} + \int_{\frac{1}{2}\pi+\pi\tau}^{-\frac{1}{2}\pi+\pi\tau} + \int_{-\frac{1}{2}\pi+\pi\tau}^{-\frac{1}{2}\pi} f(z)e^{2niz}\,dz.$$

Since $f(z+\pi) = f(z)$ and $e^{2ni(z+\pi)} = e^{2niz}$, the second and fourth integrals are equal and of opposite sign, and therefore they cancel in the preceding sum:

$$\begin{aligned}\int_C f(z)e^{2niz}\,dz &= \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} f(z)e^{2niz}\,dz + \int_{\frac{1}{2}\pi+\pi\tau}^{-\frac{1}{2}\pi+\pi\tau} f(z)e^{2niz}\,dz \\ &= \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} f(z)e^{2niz}\,dz - \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} f(z+\pi\tau)e^{2ni(z+\pi\tau)}\,dz \\ &= (1-e^{2ni\pi\tau})\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} f(z)e^{2niz}\,dz \\ &= (1-q^{2n})\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} f(z)e^{2niz}\,dz = 2\pi i\Sigma R.\end{aligned}$$

At the point z_1 the residual of the integrand is:

$$-i\omega e^{2niz_1} = -i\omega e^{n i\frac{\pi\tau}{2}} = -iq^{\frac{n}{2}}\omega,$$

and at z_2 the residual is:

$$i\omega e^{2niz_2} = i\omega e^{n i\frac{3\pi\tau}{2}} = iq^{\frac{3n}{2}}\omega.$$

Hence:

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} f(z)e^{2niz}\,dz = \frac{2\pi i}{1-q^{2n}}\left(-iq^{\frac{n}{2}}\omega + iq^{\frac{3n}{2}}\omega\right) = 2\pi\omega\,\frac{q^{\frac{n}{2}}}{1+q^n};$$

hence:

$$a_{2n} = 4\omega\,\frac{q^{\frac{n}{2}}}{1+q^n}, \qquad n \neq 0. \tag{7}$$

2b. Derivation of the secular term a_0 for the function e^{-z}

The preceding method is not applicable to $n = 0$ because the integral of $f(z)$ along the contour previously indicated vanishes. We shall therefore find the value of the integral of a more general expression:

$$\varphi(z) = \frac{A + Bk\,\mathrm{sn}^2\left(\frac{2Kz}{\pi}\right)}{1 + k\,\mathrm{sn}^2\left(\frac{2Kz}{\pi}\right)}$$

where A and B are arbitrary constants. The residual of $\varphi(z)$ at the point $z_1 = \frac{\pi\tau}{4}$ is:

$$R = \lim_{z \to z_1} (z - z_1)\varphi(z) = \frac{A + Bk \operatorname{sn}^2\left(\frac{2Kz_1}{\pi}\right)}{\frac{d}{dz_1}\left(1 + k \operatorname{sn}^2\left(\frac{2Kz_1}{\pi}\right)\right)}$$

$$= \frac{(A - B)\pi}{2k \cdot 2Ki \frac{1 + k}{k}} = -i\omega \frac{A - B}{\alpha - \delta}.$$

Taking the integral of the expression $\varphi(z)$ along the contour C_1: $\left(-\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2} + \frac{\pi\tau}{2}, -\frac{\pi}{2} + \frac{\pi\tau}{2}, -\frac{\pi}{2}\right)$ of Figure 1,

$$\int_{C_1} \varphi(z)\, dz = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} + \int_{\frac{1}{2}\pi}^{\frac{1}{2}\pi + \frac{1}{2}\pi\tau} + \int_{\frac{1}{2}\pi + \frac{1}{2}\pi\tau}^{-\frac{1}{2}\pi + \frac{1}{2}\pi\tau} + \int_{-\frac{1}{2}\pi + \frac{1}{2}\pi\tau}^{-\frac{1}{2}\pi} \varphi(z)\, dz;$$

the second and fourth integrals cancel, as in the previous case. The only pole within the contour is the point z_1.

Bearing in mind the relation:[13]

$$\operatorname{sn}\,(u + iK') = \frac{1}{k} \operatorname{ns}\, u$$

we may write

$$\int_{C_1} \varphi(z)\, dz = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \varphi(z)\, dz - \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \varphi\left(z + \frac{\pi\tau}{2}\right) dz$$

$$= \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{A + Bk \operatorname{sn}^2\left(\frac{2Kz}{\pi}\right)}{1 + k \operatorname{sn}^2\left(\frac{2Kz}{\pi}\right)} dz - \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{A + Bk \operatorname{sn}^2\left(\frac{2Kz}{\pi} + iK'\right)}{1 + k \operatorname{sn}^2\left(\frac{2Kz}{\pi} + iK'\right)} dz$$

$$= \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{A + Bk \operatorname{sn}^2\left(\frac{2Kz}{\pi}\right)}{1 + k \operatorname{sn}^2\left(\frac{2Kz}{\pi}\right)} dz - \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{Ak \operatorname{sn}^2\left(\frac{2Kz}{\pi}\right) + B}{1 + k \operatorname{sn}^2\left(\frac{2Kz}{\pi}\right)} dz$$

$$= I_1 - I_2 = 2\pi i \Sigma R = 2\pi\omega \frac{A - B}{\alpha - \delta}.$$

[13] Whittaker and Watson, loc. cit., par. 22.34.

On the other hand:

$$I_1 + I_2 = \int_{-\frac{1}{2}\tau}^{\frac{1}{2}\tau} \frac{A + Bk\,\mathrm{sn}^2\left(\frac{2Kz}{\pi}\right)}{1 + k\,\mathrm{sn}^2\left(\frac{2Kz}{\pi}\right)}\,dz + \int_{-\frac{1}{2}\tau}^{\frac{1}{2}\tau} \frac{Ak\,\mathrm{sn}^2\left(\frac{2Kz}{\pi}\right) + B}{1 + k\,\mathrm{sn}^2\left(\frac{2Kz}{\pi}\right)}\,dz$$

$$= \int_{-\frac{1}{2}\tau}^{\frac{1}{2}\tau} (A + B)\,dz = \pi(A + B).$$

Therefore

$$I_1 = \frac{1}{2}\left(\pi(A + B) + 2\pi\omega\,\frac{A - B}{\alpha - \delta}\right) = \frac{\pi}{2}\left(A + B + 2\omega\,\frac{A - B}{\alpha - \delta}\right). \tag{8}$$

Hence, to derive the secular term a_0 of the function e^{-z}, it is sufficient to place in the previous results $A = \alpha$ and $B = \delta$, and we obtain

$$a_0 = \frac{2}{\pi}\int_{-\frac{1}{2}\tau}^{\frac{1}{2}\tau} \frac{\alpha + \delta k\,\mathrm{sn}^2\left(\frac{2Kz}{\pi}\right)}{1 + k\,\mathrm{sn}^2\left(\frac{2Kz}{\pi}\right)}\,dz$$

$$= \frac{2}{\pi}\,I_1 = 1 + 2\omega. \tag{9}$$

2c. The Fourier coefficients of the function e^{-2z}

For the function e^{-2z} we find:

$z_1 = \frac{\pi\tau}{4}$, double pole with its principal part $-\frac{\omega^2}{(z - z_1)^2} - \frac{i\omega}{z - z_1}$,

$z_2 = \frac{3\pi\tau}{4}$, double pole with its principal part $-\frac{\omega^2}{(z - z_2)^2} + \frac{i\omega}{z - z_2}$,

which are the only singular points within the cell of Figure 1.

A method similar to that used in the previous case leads to the following:

$$a_0 = 2\omega\,\frac{\alpha - \delta}{1 + k}\,\frac{2E}{\pi} + \frac{k^2}{1 + k^2} + 2\omega, \tag{10}$$

$$a_{2n} = \frac{4\omega q^{\frac{n}{2}}}{1 + q^n} + 4n\omega^2\,\frac{2q^{\frac{n}{2}}}{1 - q^n}, \qquad (n \neq 0). \tag{11}$$

2d. The Fourier coefficients for the function e^{2z}

For the function e^{2z} the singular points are:

$z_1 = \frac{\pi\tau}{2} + r$, double pole with principal part $\frac{\omega^2}{a(z - z_1)^2}$,

$z_2 = \frac{\pi\tau}{2} - r$, double pole with principal part $\frac{\omega^2}{a(z - z_2)^2}$,

where r satisfies the relation

$$\operatorname{sn} \frac{2Kr}{\pi} = \sqrt{-\delta/\alpha k},$$

or its equivalent

$$r = \frac{\pi}{2K} \int_0^{\sqrt{-\delta/\alpha k}} \frac{dz}{\sqrt{(1 - z^2)(1 - k^2 z^2)}}.$$

The corresponding Fourier series is given by the formulas:

$$a_0 = \frac{1}{a}\left(\frac{1}{1 + k^2} - 2\omega \frac{\alpha - \delta}{1 + k} \frac{2E}{\pi}\right), \tag{12}$$

$$a_{2n} = -\frac{16q^n}{1 - q^{2n}} \frac{\omega^2 n}{a} \cos(2nr), \qquad (n \neq 0). \tag{13}$$

2e. Extension of the residual theorem for a particular case of multiform functions

The residual theorem is not applicable to multiform functions. To obtain a similar expression for this kind of functions, we make, from the starting point of the integration, cuts to the branch points[14] of the function within the contour

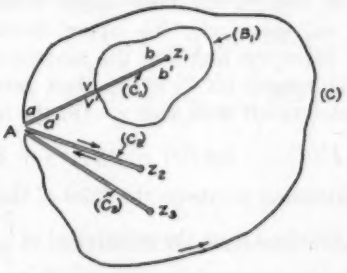

FIG. 2. Illustrating method of cuts

going around these points by means of circles of small radius connected with the cuts (Fig. 2). Connecting these cuts and circles which surround the branch points to the contour C, a simple connected region is obtained in which each of the branches of the function is a uniform function, to which the residual theorem then applies. If the multiform function is of the type $F'(z) \log f(z)$, where $F'(z)$ as well as $f(z)$ are meromorphic functions[10] within the contour of integration, the branch points of the previous expression are the zeros and poles of the

[14] Appell and Goursat, "Théorie des Fonctions Algébriques," Paris, Gauthier-Villars, 2nd Ed., 1929. Vol. I, Chapter IV. The properties of algebraic multiform functions with their singularities and methods for their study are given in this book. Their methods can be easily extended to all multiform functions, in particular the notion of branch points and the method of transformation of multiform into uniform functions either by the method of cuts or by the method of Riemann surfaces. The method of cuts is frequently used to study the integrals of algebraic functions, a problem similar to that dealt with here.

function $f(z)$. Assuming that the poles of the function $F'(z)$ do not coincide with the branch points, it is possible to find an analytical expression for the integral of the expression in question along a closed contour. If the cut and the circle corresponding to a branch point are denoted by C_n, the residual theorem may be written:

$$\int_{C(A)} F'(z) \log f(z)\, dz + \int_{C_1} + \int_{C_2} + \cdots = 2\pi i \Sigma R \tag{14}$$

where $\int_{C(A)}$ indicates that the integral is carried out starting from the point A and the R's are the residuals of the integrand within the contour and can only arise from the poles of $F'(z)$. If z_1 is a zero of order n_1 of the function $f(z)$, we shall have:

$$\int_{C_1} F'(z) \log f(z)\, dz = \int_a^b + \int_{b'}^{a'} + \int_{C_1'} F'(z) \log f(z)\, dz,$$

where C_1' is the circle surrounding the first branch point z_1. In the vicinity of the point z_1, the function $f(z)$ may be written $f(z) = (z - z_1)^{n_1}\varphi(z)$, where $\varphi(z)$ is regular for $z = z_1$ and does not vanish. Taking the circle C_1' of radius r and denoting, by M and m respectively, the upper bounds of $|F'(z)|$ and $|\log \varphi(z)|$ within and on the circle, the upper bound of the integrand is $M(m + n_1 |\log r|)$. Since the length of the contour of integration is $2\pi r$, the absolute value of the integral on C_1' is less than $2\pi r M(m + n_1 |\log r|)$, an expression which tends toward zero with r. On the other hand, we have:

$$F'(v) = F'(v'); \qquad \log f(v) = \log f(v') + 2n_1\pi i,$$

where v and v' are corresponding points on the sides of the cut.

The second relation is obtained from the integration of $\dfrac{f'(z)}{f(z)}$ along the contour B_1 (Fig. 2) which includes only one branch point z_1. Since z_1 is the only singular point of this function with a residual n_1, the previous equation is the result of the application of the residual theorem.[15] Therefore,

$$\int_a^b F'(z) \log f(z)\, dz + \int_{b'}^{a'} F'(z) \log f(z)\, dz$$

$$= \int_a^b - \int_{a'}^{b'} F'(z) \log f(z)\, dz = \int_a^b 2n_1\pi i F'(z)\, dz$$

$$= 2n_1\pi i[F(b) - F(a)].$$

If the radius r of the circle surrounding z_1 tends to zero, we obtain the relation:

$$\int_{C_1} F'(z) \log f(z)\, dz = 2n_1\pi i[F(z_1) - F(a)].$$

In the case of a pole, the expression obtained is similar but has the opposite sign.

[15] E. Goursat, loc. cit., par. 299.

Substituting this and similar expressions in equation (14), we obtain:

$$\int_{C(A)} F'(z) \log f(z)\, dz$$
$$= 2\pi i[-\sum_c n_c F(z_c) + \sum_p n_p F(z_p) + (\sum_c n_c - \sum_p n_p)F(a) + \sum R], \quad (15)$$

where n_c is the order of the zero z_c, and n_p that of the pole z_p. It may be noted that the integral is independent of the initial point only in the case when the number of poles and of zeros within the contour of integration, taking account of the order of multiplicity of each, are the same.

2f. Derivation of the Fourier coefficients $a_{2n} (n \neq 0)$ for the function x

To find the coefficients a_{2n} of the Fourier expansion of the function

$$x = \log \frac{1 + k \operatorname{sn}^2\left(\frac{2Kz}{\pi}\right)}{\alpha + \delta k \operatorname{sn}^2\left(\frac{2Kz}{\pi}\right)},$$

we substitute in (15) in place of $f(z)$:

$$e^x = \frac{1 + k \operatorname{sn}^2\left(\frac{2Kz}{\pi}\right)}{\alpha + \delta k \operatorname{sn}^2\left(\frac{2Kz}{\pi}\right)}$$

and for $F'(z)$:

$$e^{2niz},$$

taking as our contour of integration the same one which was used for the other functions; i.e., C: $\left(-\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2} + \pi\tau, -\frac{\pi}{2} + \pi\tau, -\frac{\pi}{2}\right)$, as shown in Figure 3.

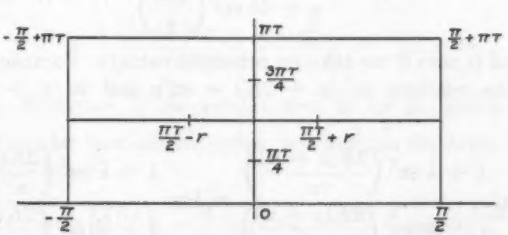

FIG. 3. Periodicity cell in z-plane of the function $\log e^x$

The function e^x has two simple zeros within the contour C, $z = \frac{\pi\tau}{4}, \frac{3\pi\tau}{4}$, and two simple poles, $z = \frac{\pi\tau}{2} \pm r$.

Since $F'(z)$ is an entire function, the sum ΣR vanishes. Likewise, the sum depending on the starting point vanishes, because the number of zeros and of poles of the function e^z is the same within the periodicity cell, this being a general property of all elliptic functions.[16] On the other hand,

$$F(z) = -\frac{i}{2n} e^{2niz}.$$

Substituting the indicated values in formula (15), we obtain:

$$\int_{C\left(-\frac{\pi}{2}\right)} \log \frac{1 + k \operatorname{sn}^2\left(\frac{2Kz}{\pi}\right)}{\alpha + \delta k \operatorname{sn}^2\left(\frac{2Kz}{\pi}\right)} e^{2niz}\, dz$$

$$= 2\pi i \left(\frac{-i}{2n}\right)\left[-e^{2ni\frac{\pi\tau}{4}} - e^{2ni\frac{3\pi\tau}{4}} + e^{2ni\left(\frac{\pi\tau}{2}+\tau\right)} + e^{2ni\left(\frac{\pi\tau}{2}-\tau\right)}\right]$$

$$= \frac{\pi}{n}\left(-q^{\frac{n}{2}} - q^{\frac{3n}{2}} + 2q^n \cos 2n\tau\right).$$

On the other hand, along the contour C, $sn^2\left(\frac{2Kz}{\pi}\right)$ remains positive and less than $1/k^2$, so that the function

$$\frac{1 + k \operatorname{sn}^2\left(\frac{2Kz}{\pi}\right)}{\alpha + \delta k \operatorname{sn}^2\left(\frac{2Kz}{\pi}\right)}$$

remains real and positive. The imaginary part of the multiform function

$$\log \frac{1 + k \operatorname{sn}^2\left(\frac{2Kz}{\pi}\right)}{\alpha + \delta k \operatorname{sn}^2\left(\frac{2Kz}{\pi}\right)}$$

is constant (and is zero if we take its principal value). Because of this, taking into account the relations $\operatorname{sn}^2(u + 2K) = \operatorname{sn}^2 u$ and $\operatorname{sn}(u + 2iK') = \operatorname{sn} u$, we may write

$$\log \frac{1 + k \operatorname{sn}^2\left(\frac{2K(z+\pi)}{\pi}\right)}{\alpha + \delta k \operatorname{sn}^2\left(\frac{2K(z+\pi)}{\pi}\right)} = \log \frac{1 + k \operatorname{sn}^2\left(\frac{2Kz}{\pi}\right)}{\alpha + \delta k \operatorname{sn}^2\left(\frac{2Kz}{\pi}\right)}$$

$$\log \frac{1 + k \operatorname{sn}^2\left(\frac{2K(z+\pi\tau)}{\pi}\right)}{\alpha + \delta k \operatorname{sn}^2\left(\frac{2K(z+\pi\tau)}{\pi}\right)} = \log \frac{1 + k \operatorname{sn}^2\left(\frac{2Kz}{\pi}\right)}{\alpha + \delta k \operatorname{sn}^2\left(\frac{2Kz}{\pi}\right)}.$$

[16] Whittaker and Watson, loc. cit., par. 20.13.

Therefore:

$$\int_{c\left(-\frac{\pi}{2}\right)} \log \frac{1 + k \operatorname{sn}^2\left(\frac{2Kz}{\pi}\right)}{\alpha + \delta k \operatorname{sn}^2\left(\frac{2Kz}{\pi}\right)} e^{2niz}\, dz$$

$$= \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} + \int_{\frac{1}{2}\pi}^{\frac{1}{2}\pi+\pi i} + \int_{\frac{1}{2}\pi+\pi i}^{-\frac{1}{2}\pi+\pi i} + \int_{-\frac{1}{2}\pi+\pi i}^{-\frac{1}{2}\pi} \log \frac{1 + k \operatorname{sn}^2\left(\frac{2Kz}{\pi}\right)}{\alpha + \delta k \operatorname{sn}^2\left(\frac{2Kz}{\pi}\right)} e^{2niz}\, dz.$$

By virtue of the relations previously pointed out and of the periodicity of the function e^{2niz}, we infer, following the same method as in previous cases, that the sum of the second and fourth integrals vanishes, and the third integral is equal to $-q^{2n}$ times the first. Therefore:

$$\int_{c\left(-\frac{\pi}{2}\right)} \log \frac{1 + k \operatorname{sn}^2\left(\frac{2Kz}{\pi}\right)}{\alpha + \delta k \operatorname{sn}^2\left(\frac{2Kz}{\pi}\right)} e^{2niz}\, dz$$

$$= (1 - q^{2n}) \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \log \frac{1 + k \operatorname{sn}^2\left(\frac{2Kz}{\pi}\right)}{\alpha + \delta k \operatorname{sn}^2\left(\frac{2Kz}{\pi}\right)} e^{2niz}\, dz.$$

$$a_{2n} = \frac{2}{\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \log \frac{1 + k \operatorname{sn}^2\left(\frac{2Kz}{\pi}\right)}{\alpha + \delta k \operatorname{sn}^2\left(\frac{2Kz}{\pi}\right)} \cos 2nz\, dz$$

$$= \frac{2}{\pi} \frac{1}{1 - q^{2n}} \frac{\pi}{n} (-q^{\frac{n}{2}} - q^{\frac{3n}{2}} + 2q^n \cos 2nr)$$

$$= \frac{4}{n} \frac{q^n \cos 2nr}{1 - q^{2n}} - \frac{2}{n} \frac{q^{\frac{n}{2}}}{1 - q^n}, \qquad (n \neq 0). \tag{16}$$

2g. Derivation of the secular term a_0 for the function x

To find the secular term of this series, we shall use the same formula, taking for $F'(z)$ the function[17]

$$Z(u) = Z\left(\frac{2Kz}{\pi}\right) = \frac{d}{du} \log \Theta(u) = \frac{\pi}{2K} \frac{d}{dz} \log \Theta\left(\frac{2Kz}{\pi}\right).$$

The function $\log \dfrac{1 + k \operatorname{sn}^2\left(\frac{2Kz}{\pi}\right)}{\alpha + \delta k \operatorname{sn}^2\left(\frac{2Kz}{\pi}\right)}$ has already been studied.

[17] Whittaker and Watson, loc. cit., par. 22.731.

The function $Z(u)$ has the properties:

$$Z(u + 2K) = Z(u), \qquad Z(u + 2iK') = Z(u) - \frac{\pi i}{K}. \tag{17}$$

It has a simple pole within the cell, $u = iK'$, with a residual equal to unity. In the z plane its pole is $z_1 = \frac{\pi\tau}{2}$ with a residual $\frac{\pi}{2K}$.

Integrating along contour C: $\left(-\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2} + \pi\tau, -\frac{\pi}{2} + \pi\tau, -\frac{\pi}{2}\right)$, we obtain, substituting in equation (15) instead of the functions $f(z)$ and $F'(z)$, e^z and $Z\left(\frac{2Kz}{\pi}\right)$:

$$\int_{c\left(-\frac{\pi}{2}\right)} \log \frac{1 + k\,\mathrm{sn}^2\left(\frac{2Kz}{\pi}\right)}{\alpha + \delta k\,\mathrm{sn}^2\left(\frac{2Kz}{\pi}\right)} Z\left(\frac{2Kz}{\pi}\right) dz$$

$$= \frac{\pi^2 i}{K} \log \frac{\Theta\left(iK' + \frac{2Kr}{\pi}\right)\cdot\Theta\left(iK' - \frac{2Kr}{\pi}\right)\cdot\frac{1}{\delta}}{\Theta\left(\frac{iK'}{2}\right)\cdot\Theta\left(\frac{3iK'}{2}\right)}$$

$$= \frac{\pi^2 i}{K} \log \frac{H^2\left(\frac{2Kr}{\pi}\right)}{(-\delta)\Theta^2\left(\frac{iK'}{2}\right)}.$$

On the other hand, bearing in mind relations (17), and following the same method as before, we may write:

$$\int_{c\left(-\frac{\pi}{2}\right)} \log \frac{1 + k\,\mathrm{sn}^2\left(\frac{2Kz}{\pi}\right)}{\alpha + \delta k\,\mathrm{sn}^2\left(\frac{2Kz}{\pi}\right)} Z\left(\frac{2Kz}{\pi}\right) dz$$

$$= \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} + \int_{\frac{1}{2}\pi}^{\frac{1}{2}\pi+\pi\tau} + \int_{\frac{1}{2}\pi+\pi\tau}^{-\frac{1}{2}\pi+\pi\tau} + \int_{-\frac{1}{2}\pi+\pi\tau}^{-\frac{1}{2}\pi} \log \frac{1 + k\,\mathrm{sn}^2\left(\frac{2Kz}{\pi}\right)}{\alpha + \delta k\,\mathrm{sn}^2\left(\frac{2Kz}{\pi}\right)} Z\left(\frac{2Kz}{\pi}\right) dz$$

$$= \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} + \int_{\frac{1}{2}\pi+\pi\tau}^{-\frac{1}{2}\pi+\pi\tau} \log \frac{1 + k\,\mathrm{sn}^2\left(\frac{2Kz}{\pi}\right)}{\alpha + \delta k\,\mathrm{sn}^2\left(\frac{2Kz}{\pi}\right)} Z\left(\frac{2Kz}{\pi}\right) dz$$

$$= \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \log \frac{1 + k\,\mathrm{sn}^2\left(\frac{2Kz}{\pi}\right)}{\alpha + \delta k\,\mathrm{sn}^2\left(\frac{2Kz}{\pi}\right)} \left[Z\left(\frac{2Kz}{\pi}\right) - Z\left(\frac{2Kz}{\pi} + 2iK'\right)\right] dz$$

$$= \frac{\pi i}{K} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \log \frac{1 + k \operatorname{sn}^2\left(\frac{2Kz}{\pi}\right)}{\alpha + \delta k \operatorname{sn}^2\left(\frac{2Kz}{\pi}\right)} dz$$

$$= \frac{\pi^2 i}{K} \log \frac{H^2\left(\frac{2Kr}{\pi}\right)}{(-\delta)\cdot\Theta^2\left(\frac{iK'}{\pi}\right)}.$$

Therefore

$$a_0 = \frac{2}{\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \log \frac{1 + k \operatorname{sn}^2\left(\frac{2Kz}{\pi}\right)}{\alpha + \delta k \operatorname{sn}^2\left(\frac{2Kz}{\pi}\right)} dz$$

$$= 2 \log \frac{H^2\left(\frac{2Kr}{\pi}\right)}{(-\delta)\cdot\Theta^2\left(\frac{iK'}{2}\right)} \tag{18}$$

2h. Fourier expansions obtained

Bearing in mind the results of the previous paragraphs, the desired Fourier series may be written down. Using the same notation for the Fourier coefficients as that suggested by A. Baños Jr.[18] we have:

$$\left.\begin{aligned} x &= z_0 + z_2 \cos 2\omega\sigma + z_4 \cos 4\omega\sigma + \cdots, \\ z_0 &= \log \frac{H^2\left(\frac{2Kr}{\pi}\right)}{(-\delta)\Theta^2\left(\frac{iK'}{2}\right)}, \\ z_{2n} &= \frac{1}{n}\left(\frac{4q^n \cos 2nr}{1 - q^{2n}} - \frac{2q^{\frac{n}{2}}}{1 - q^n}\right). \end{aligned}\right\} \tag{19}$$

For $X = ae^{2x} - e^{-x} + e^{-2x}$, we have:

$$\left.\begin{aligned} X &= Z_0 + Z_2 \cos 2\omega\sigma + Z_4 \cos 4\omega\sigma + \cdots, \\ Z_0 &= 0, \\ Z_{2n} &= 4n\omega^2\left(-\frac{4q^n \cos 2nr}{1 - q^{2n}} + \frac{2q^{\frac{n}{2}}}{1 - q^n}\right) = -4n^2\omega^2 z_{2n}. \end{aligned}\right\} \tag{20}$$

Equation (2) may be verified immediately by taking the values of the coefficients Z_{2n} and z_{2n}.

[18] Alfredo Baños Jr., Journal of Mathematics and Physics, Vol. XVIII, No. 3, p. 211 (1939).

In order to give to equations (4) and (5) the form of Hill's equation, we shall write all the terms in the first member and divide through by ω^2. The coefficients of the two series of the equations in ψ_1 and η_1 shall be denoted by A and B, because they are connected with the coefficients a and b, discussed by A. Baños Jr.,[18] by a constant coefficient. Hill's coefficients corresponding to equations (4) and (5) have the following expansions:

$$-\frac{1}{\omega^2}X_x = -\frac{1}{\omega^2}(2ae^{2x} - 2e^{-2x} + e^{-x}) = A_0 + 2A_2\cos 2\omega\sigma + 2A_4\cos 4\omega\sigma + \cdots,$$

$$-\frac{1}{\omega^2}\Lambda_\lambda = -\frac{1}{\omega^2}(e^{-2x} - 1) = B_0 + 2B_2\cos 2\omega\sigma + 2B_4\cos 4\omega\sigma + \cdots.$$

The values of these coefficients are given by the expressions:

$$\left.\begin{aligned} A_0 &= -\frac{1}{2\omega^2}\left(\frac{3-k^2}{1+k^2} - 2\omega - 8\omega\frac{\alpha-\delta}{1+k}\frac{2E}{\pi}\right), \\ A_{2n} &= \frac{16nq^n\cos 2nr}{1-q^{2n}} + \frac{8nq^{\frac{n}{2}}}{1-q^n} + \frac{2q^{\frac{n}{2}}}{\omega(1+q^n)}, \end{aligned}\right\} \tag{21}$$

$$\left.\begin{aligned} B_0 &= \frac{1}{2\omega^2}\left(\frac{2+k^2}{1+k^2} - 2\omega\frac{\alpha-\delta}{1+k}\frac{2E}{\pi} - 2\omega\right), \\ B_{2n} &= -\frac{4nq^{\frac{n}{2}}}{1-q^n} - \frac{2q^{\frac{n}{2}}}{\omega(1+q^n)}. \end{aligned}\right\} \tag{21a}$$

The formation of the coefficients of the series for X, $\frac{1}{\omega^2}X_x$ and $\frac{1}{\omega^2}\Lambda_\lambda$, by applying the commutative and associative laws to the sum, and differentiating the series for x term by term to compare it with that for X, is justified because the series for x, e^{-x}, e^{-2x} and e^{2x} are uniformly and absolutely convergent, as well as the series formed by the second derivatives of the series for $x = x(\sigma)$. The proof is easy if we note that:

$$\lim_{n\to\infty}\frac{z_{2n+2}}{z_{2n}} = q^{\frac{1}{2}} < 1,$$

and therefore the coefficients of the trigonometrical series for x are less than those corresponding to a convergent geometrical series of constant terms.

For the series derived from that which represents $x = x(\sigma)$ by differentiation term by term, the ratio of each coefficient to that immediately preceding it also has the limit $q^{\frac{1}{2}} < 1$, and therefore all these series are uniformly and absolutely convergent for real values of σ.

If the imaginary part of $\omega\sigma$ is less in absolute value than $p < -i\frac{\pi\tau}{4}$, the expression

$$\overline{\lim_{n\to\infty}}\,|z_{2n}\cos 2n\omega\sigma|^{\frac{1}{n}} \leq q^{\frac{1}{2}}e^{2p} = e^{i\frac{\pi\tau}{2}+2p} < 1.$$

Within the strip $-p \leq I(\omega\sigma) \leq p$, the series obtained is uniformly convergent because its terms are less in absolute value than those of a convergent geometric series of constant terms. Within the specified region, the trigonometrical series for x will be an analytical function of σ.[19]

A similar argument shows that the functions e^{-z} and e^{-2z} defined by trigonometrical series are analytical functions within the same region.

For the function e^{2z} the convergence region is wider because

$$\overline{\lim} \mid a_{2n} \mid^{\frac{1}{n}} = q.$$

Applying the same arguments as for the other series, we obtain for the region where the function is analytical:

$$-p_1 \leq I(\omega\sigma) \leq p_1 < -i\frac{\pi\tau}{2}.$$

The preceding results are in agreement with the distribution of the singular points of the functions we have studied.

2i. Numerical verification of the formulas of the preceding paragraph

In order to verify the formulas which have been obtained, we have calculated the parameters and the Fourier coefficients corresponding to the orbit $q = 0.01$ ($\gamma_1 = 1.221{,}6842$) to seven decimals, applying the methods in practical use[20] for Fourier analysis and comparing them with the results obtained from the formulas. The maximum discrepancy is not greater than two units of the seventh decimal, which is considered to be a check on the correctness of the formulas we have obtained.

3. Results of the Numerical Calculations for Different Values of the Parameter q

3a. Tabulation of the numerical results

In order to obtain a general idea of the behavior of the orbits, when q varies from zero to one, we have calculated eight orbits for values of $q(0 < q < 1)$, arbitrarily chosen. Tables I–V show the results of these calculations. In the first line the numbers $2, 3, \cdots, 9$ indicate the orbits calculated, and the numbers 1 and 10 are reserved for the limiting cases $q = 0$ and $q = 1$, respectively. The corresponding values of q and of γ_1 appear in the second and third lines. Table I contains the principal parameters which we have as a basis for the calculation; Table II contains the Fourier coefficients of the abscissa x; Table III the Fourier coefficients of the acceleration $\ddot{x} = X$; and Tables IV and V the Fourier coefficients of Hill's equations (4) and (5), the explicit values of which are given by equations (21) and (21a).

The values shown in the tables have been calculated first by means of our formulas and then by means of a numerical calculation, in order to verify the

[19] E. Goursat, loc. cit., par. 290.

[20] Whittaker and Robinson, "The Calculus of Observations." Blackie and Son, Ltd., London, 2nd ed., 1932.

TABLE I

	1	2	3	4	5	6	7	8	9	10
q	0	0.00101,7	0.00241,3	0.00465,6	0.00821,7	0.01423,8	0.02666,8	0.06887,3	0.23383,8	1
γ_1	∞	2.00000	1.63299	1.41421	1.26491	1.15470	1.06905	1.01000	1.00002	1
k	0	0.12702	0.19460	0.26795	0.35100	0.45142	0.58957	0.81860	0.99106	1
α	1	1.05902	1.08630	1.11237	1.13738	1.16144	1.18465	1.20362	1.20710	1.20711
β	1	0.93301	0.89528	0.85355	0.80619	0.75000	0.67678	0.57019	0.50316	0.50000
γ	0	0.06699	0.10472	0.14645	0.19381	0.25000	0.32322	0.42981	0.49684	0.50000
σ	0	−.05902	−.08630	−.11237	−.13738	−.16144	−.18465	−.20362	−.20710	−.20711
$\frac{2K}{\pi}$	1	1.00407	1.00968	1.01871	1.03314	1.05776	1.10952	1.29457	2.17167	∞
$\frac{2E}{\pi}$	1	0.99595	0.99046	0.98181	0.96845	0.94692	0.90642	0.80164	0.65316	0.63662
ω	0.5	0.49400	0.48609	0.47409	0.45665	0.43083	0.38820	0.29887	0.16553	0
r	45°	41°22′30″	39°26′50″	37°22′00″	35°01′40″	32°13′20″	28°21′10″	21°25′05″	11°35′40″	0

TABLE II

	1	2	3	4	5	6	7	8	9	10
q	0	101,7	241,3	465,6	821,7	1423,8	2666,8	6887,3	23383,8	1
γ_1	∞	2.00000	1.63299	1.41421	1.26491	1.15470	1.06905	1.01000	1.00002	1
z_0	0	702	1634	3063	5171	8361	13810	25923	45433	69315
z_2		−6332	−9662	−13221	−17158	−21753	−27695	−36071	−35394	0
z_4		−102	−242	−469	−832	−1450	−2725	−6849	−17216	0
z_6		−2	−8	−21	−50	−114	−293	−1233	−7055	0
z_8				−1	−3	−10	−35	−239	−2760	0
z_{10}						−1	−5	−50	−1083	0
z_{12}							−1	−11	−435	0
z_{14}								−3	−179	0
z_{16}								−1	−75	0
z_{18}									−33	0
z_{20}									−14	0
z_{22}									−7	0
z_{24}									−3	0
z_{26}									−1	0

TABLE III

	1	2	3	4	5	6	7	8	9	10
q	0	101,7	241,3	465,6	821,7	1423,8	2666,8	6887,3	23383,8	1
γ_1	∞	2.00000	1.63299	1.41421	1.26491	1.15470	1.06905	1.01000	1.00002	1
Z_2	0	6181	9132	11886	14312	16151	16695	12888	8786	0
Z_4		398	916	1688	2776	4305	6571	9787	7366	0
Z_6		19	67	172	373	859	1589	3963	6792	0
Z_8		1	4	16	45	121	344	1369	4724	0
Z_{10}				1	5	18	73	446	2899	0
Z_{12}					1	3	14	140	1675	0
Z_{14}							3	43	938	0
Z_{16}								14	515	0
Z_{18}								4	278	0
Z_{20}									150	0
Z_{22}									80	0
Z_{24}									42	0
Z_{26}									22	0
Z_{28}									11	0
Z_{30}									6	0
Z_{32}									3	0
Z_{34}									1	0

TABLE IV

	1	2	3	4	5	6	7	8	9	10
q	0	101,7	241,3	465,6	821,7	1423,8	2666,8	6887,3	23383,8	1
	∞	2.00000	1.63299	1.41421	1.26491	1.15470	1.06905	1.01000	1.00002	1
A_0	4.0000	4.0079	4.0182	4.0331	4.0529	4.0769	4.0911	3.8388	−1.5702	$-\infty$
A_2	0	3863	6030	8546	1.1698	1.6128	2.3961	4.7098	13.4762	∞
A_4	0	203	484	935	1658	2899	5552	1.5774	7.8731	∞
A_6		9	33	90	211	485	1261	5451	4.3238	∞
A_8			2	8	25	74	264	1821	2.4104	∞
A_{10}				1	3	11	54	580	1.3563	∞
A_{12}						2	10	179	7582	∞
A_{14}							2	54	4178	∞
A_{16}								16	2268	∞
A_{18}								5	1215	∞
A_{20}								1	644	∞
A_{22}									339	∞
A_{24}									177	∞
A_{26}									92	∞
A_{28}									47	∞
A_{30}									24	∞
A_{32}									12	∞
A_{34}									6	∞
A_{36}									3	∞

TABLE V

	1	2	3	4	5	6	7	8	9	10
q	0	101,7	241,3	465,6	821,7	1423,8	2666,8	6887,3	23383,8	1
γ_1	∞	2.00000	1.63299	1.41421	1.26491	1.15470	1.06905	1.01000	1.00002	1
B_0	0	409	977	1905	3415	6071	1.1926	3.5281	19.1775	∞
B_2		−2566	−3986	−5608	−7593	−1.0303	−1.4906	−2.7705	−7.3180	$-\infty$
B_4		−122	−292	−569	−1017	−1800	−3508	−1.0123	−4.6907	$-\infty$
B_6		−5	−19	−52	−122	−283	−747	−3379	−2.7400	$-\infty$
B_8			−1	−4	−14	−42	−151	−1076	−1.5444	$-\infty$
B_{10}					−1	−6	−30	−332	−8524	$-\infty$
B_{12}						−1	−6	−100	−4635	$-\infty$
B_{14}							−1	−30	−2488	$-\infty$
B_{16}								−10	−1323	$-\infty$
B_{18}								−3	−698	$-\infty$
B_{20}									−365	$-\infty$
B_{22}									−190	$-\infty$
B_{24}									−99	$-\infty$
B_{26}									−51	$-\infty$
B_{28}									−26	$-\infty$
B_{30}									−13	$-\infty$
B_{32}									−7	$-\infty$
B_{34}									−3	$-\infty$
B_{36}									−1	$-\infty$

results. For orbits 1 and 10, many of the values cannot be obtained by means of the formulas, but instead have been found by a passage to the limit. For instance, to find the value of r for orbit 1, we may note that

$$\lim_{q\to 0}\frac{-\delta}{\alpha k}=\lim_{q\to 0}\frac{\sqrt{1+1/\gamma_1^2}-1}{\sqrt{1+1/\gamma_1^2}+1}\cdot\frac{\sqrt{1+1/\gamma_1^2}+\sqrt{1-1/\gamma_1^2}}{\sqrt{1+1/\gamma_1^2}-\sqrt{1-1/\gamma_1^2}}$$

$$=\lim_{q\to 0}\left[\frac{\sqrt{1+1/\gamma_1^2}+\sqrt{1-1/\gamma_1^2}}{\sqrt{1+1/\gamma_1^2}+\quad 1}\right]^2\cdot\frac{1+1/\gamma_1^2-1}{1+1/\gamma_1^2-(1-1/\gamma_1^2)}$$

$$=\frac{1}{2}\lim_{q\to 0}\left[\frac{\sqrt{1+1/\gamma_1^2}+\sqrt{1-1/\gamma_1^2}}{\sqrt{1+1/\gamma_1^2}+\quad 1}\right]^2$$

$$=\frac{1}{2}\lim_{\gamma_1\to\infty}\left[\frac{\sqrt{1+1/\gamma_1^2}+\sqrt{1-1/\gamma_1^2}}{\sqrt{1+1/\gamma_1^2}+\quad 1}\right]^2=\frac{1}{2}.$$

The parameter r is defined by the equation:

$$\operatorname{sn}\frac{2Kr}{\pi}=\sqrt{\frac{-\delta}{\alpha k}},$$

or its equivalent:

$$\frac{2Kr}{\pi}=\int_0^{\sqrt{-\delta/\alpha k}}\frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}.$$

Bearing in mind the inequalities:

$$\frac{1}{\sqrt{1-z^2}}\leq\frac{1}{\sqrt{(1-z^2)(1-k^2z^2)}}\leq\frac{1}{\sqrt{(1-z^2)(1-k^2)}}$$

and integrating between the limits 0 and $\sqrt{-\delta/\alpha k}$ we obtain:

$$\sin^{-1}\sqrt{\frac{-\delta}{\alpha k}}\leq\frac{2Kr}{\pi}\leq\frac{1}{\sqrt{1-k^2}}\sin^{-1}\sqrt{\frac{-\delta}{\alpha k}}.$$

When q tends toward 0, $\dfrac{2Kr}{\pi}$ tends toward $\sin^{-1}\sqrt{\dfrac{1}{2}}=\dfrac{\pi}{4}=45°$, because it lies within two bounds which tend toward the same common limit. On the other hand, $\lim\limits_{q\to 0}\dfrac{2K}{\pi}=1$, and therefore $\lim\limits_{q\to 0}r=45°$.

3b. Graphical representation of the preceding numerical results

Figures 4 to 7 show graphically the more important numerical results given in Tables I, II, IV and V. In all these figures the parameter q has been taken as the independent variable. In order that the points corresponding to small values of q (γ_1 large) do not come too close together, which would be rather in-

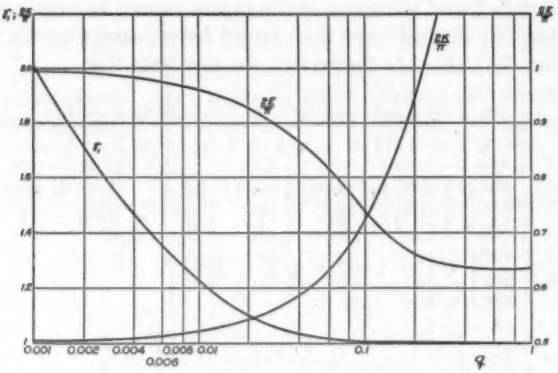

FIG. 4. Illustrating results contained in Table I

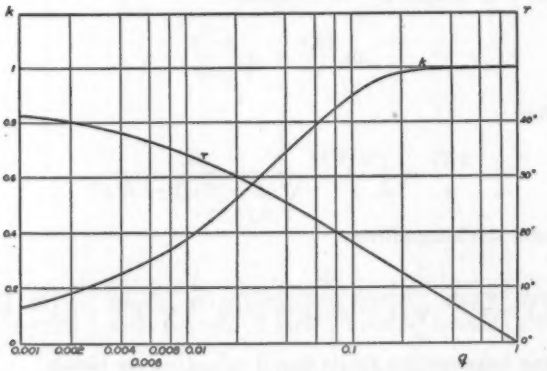

FIG. 5. Illustrating results contained in Table II

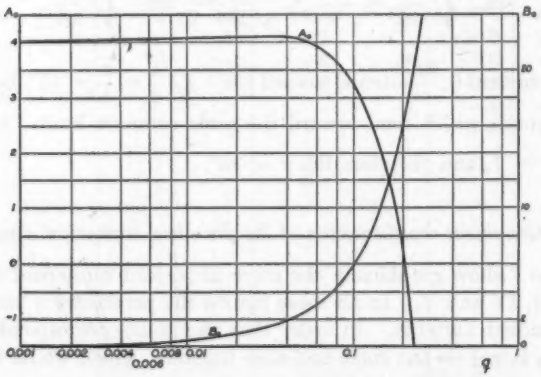

FIG. 6. Illustrating results contained in Table IV

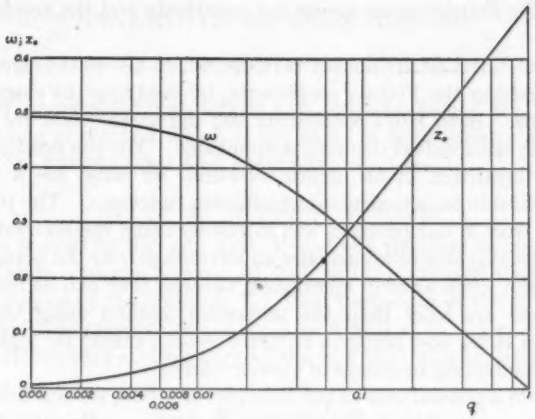

FIG. 7. Illustrating results contained in Table V

convenient because then the slope of our graphs would become too large, we have used semi-logarithmic plots by means of which the region in the vicinity of the point $q = 0$ has been extended to $-\infty$.

4. Summary

A) The study of the motion of an electrically charged particle in the field of a magnetic dipole depends on the solution of three simultaneous differential equations of the second order. Taking spherical coordinates to specify the position of the charged particle, which offer the advantage that they possess the same symmetry as the dipole, we may reduce the system of differential equations to a system of two equations of the second order, and so the study of the motion in three-dimensional space depends on the knowledge of the motion of the particle in its own meridian plane and the rotation of the meridian plane about the polar axis. The system is not directly integrable but, by means of numerical integration, interesting properties of the orbits, in particular of the periodic orbits orthogonal to the intersection of the equatorial and meridian planes, have been discovered by Störmer and studied in detail by Lemaitre and Vallarta. The orbits in the equatorial plane can be expressed in terms of elliptic functions and, in the particular case where Störmer's parameter γ_1 is greater than unity, reduce to a rectilinear periodic orbit in the meridian plane.

B) To study the symmetrical orbits, which for the value of $\gamma_1 = 1.31359$ tend toward the equatorial orbits, and to investigate in general the behavior of orbits in the meridian plane in the vicinity of equatorial orbits, it is necessary to solve the variational equations which, for the equatorial orbits, reduce to two Hill equations. The periodic function appearing in the canonical form of Hill's equation has been calculated as a Fourier series. In our own particular case, we must find the Fourier series when the corresponding elliptic functions are known. For the verification of these results, through the equation of motion,

we also find the Fourier series giving the coordinate and the acceleration of the particle.

C) All these functions are doubly periodic, which allows the calculation of the integrals expressing the Fourier coefficients, by studying the singular points of these functions. Both Hill's coefficients and the acceleration are elliptic functions, to which the residual theorem is applicable. For the position coordinate, which is the logarithm of an elliptic function, we must use a more general theorem, applicable to non-algebraic multiform functions. The particular case which is met with in our problem was solved by using the method of cuts.

D) The formulas now obtained offer an advantage over the numerical method of obtaining the series already mentioned, because they can be readily used for calculation and are safer than the numerical method using the trapezoidal formula. The latter does not give accurate results, chiefly for higher harmonics or orbits corresponding to values of γ_1 near unity.

E) To obtain a general idea of the behavior of orbits, several have been calculated, taking as our principal parameter Jacobi's q. For the limiting cases $q = 0$ and $q = 1$, which in the strict sense do not represent periodic orbits, since for $q = 0$ the orbit reduces to a point, and for $q = 1$ the period tends to infinity, special methods have been used to calculate the limiting values of some of the coefficients desired.

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TWO NOTES ON INEQUALITIES

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In the two paragraphs that follow two different topics are treated. The point in common is method which is the same in both cases. The method is not new and has already been used for similar purposes.[1]

§1. *On an inequality of Lukács*

It has been proved by Lukács[2] that, if $f(x)$ is a function of period 2π and of bounded variation over the interval $(0, 2\pi)$, and if $\sigma_n(x)$ is the n-th Fejér mean of the Fourier series of the function $f(x)$, then the length of the curve $y = \sigma_n(x)$ does not exceed the length of the curve $y = f(x)$, $0 \leq x \leq 2\pi$.

This is an elementary consequence of the fact that $\sigma_n(x)$ is equal to

$$\int_a^b K(t)f(x+t)\,dt \tag{1}$$

where $(a, b) = (0, 2\pi)$, and $K(t) = K_n(t)$ is a non negative function whose integral over $(0, 2\pi)$ is equal to 1.[3] These properties of $K(t)$ are the only ones which are needed in the proof of Lukács' theorem, so that the latter holds for general transformations of the form (1).

Still, the assumption that a linear transformation is of the form (1) is rather special and it will be shown in this § that the result stated above holds in a more general case. The proof is less elementary than that of Lukács for it uses certain theorems from the theory of probability. On the other hand, it may deserve some attention as an application of facts from the calculus of probability to problems of general Analysis. The idea used is not new.[4] Another application of it will be found in §2.

In what follows, by a *linear family* of functions defined on interval (a, b) we shall mean any set S of functions $f(x)$, $a \leq x \leq b$, such that if $f_1(x)$ and $f_2(x)$ belong to S, so does any linear combination $c_1f_1 + c_2f_2$ with constant coefficients. In order to simplify the considerations, we shall confine our attention to the case of real functions, although the argument can easily be extended to the complex case.

Theorem. *Let S be a linear family of functions $f(x)$ of bounded variation defined on an interval $a \leq x \leq b$, and let*

$$\varphi = T[f] \tag{2}$$

[1] See A. Zygmund, *On the Convergence and Summability of Power Series*, Fundamenta Math., 30 (1938), 170–196; J. Marcinkiewicz and A. Zygmund, *Quelques inégalités pour les opérations linéaires*, Fundamenta Math., 32 (1939), 115–121.

[2] See G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, I, 1925, pp. 56 and 213.

[3] See Pólya and Szegö, *loc. cit.*

[4] See the papers quoted in footnote [1].

be a transformation of functions $f(x)$ from S into functions $\varphi(\xi)$ defined on an interval $\alpha \leq \xi \leq \beta$. The only condition imposed on T is linearity[5] *and the existence of a constant M independent of f, such that*

$$\int_\alpha^\beta |\, d\varphi(\xi)\,| \leq M \int_a^b |\, df(x)\,| \tag{3}$$

for any $f \in S$.

Then, for any system of functions $f_1, f_2, \cdots, f_m$ from S, we have the inequality

$$\int_\alpha^\beta \sqrt{(d\varphi_1)^2 + (d\varphi_2)^2 + \cdots + (d\varphi_m)^2} \leq M \int_a^b \sqrt{(df_1)^2 + (df_2)^2 + \cdots + (df_m)^2} \tag{4}$$

where $\varphi_i = T[f_i]$.

The integral on the left (and similarly that on the right) of (4) is meant as the limit (and at the same time the upper bound) of the sums

$$\sum_{i=1}^{n} \left\{ \sum_{k=1}^{m} [\varphi k(\xi_i) - \varphi k(\xi_{i-1})]^2 \right\}^{\frac{1}{2}} \qquad (\alpha = \xi_0 < \xi_1 < \cdots < \xi_n = \beta)$$

as Max $(\xi_i - \xi_{i-1})$ tends to 0.

In other words, the length of the $(m+1)$-dimensional curve

$$\eta_1 = \varphi_1(\xi),\ \eta_2 = \varphi_2(\xi),\ \cdots,\ \eta_m = \varphi_m(\xi)$$

does not exceed M times the length of the $(m+1)$-dimensional curve

$$y_1 = f_1(x),\ y_2 = f_2(x),\ \cdots,\ y_m = f_m(x).$$

This is an extension to variation of a similar theorem concerning transformation of $L^p(a, b)$ to $L^p(\alpha, \beta)$. Namely, if $\varphi = T[f]$ is a linear transformation of functions $f(x)$ belonging to a linear family S of functions of the class $L^p(a, b)$ into functions $\varphi(\xi)$ of a class $L^p(\alpha, \beta)$, $p > 0$, and if in addition

$$\left\{ \int_\alpha^\beta |\, \varphi(\xi)\,|^p\, d\xi \right\}^{1/p} \leq M \left\{ \int_a^b |\, f(x)\,|^p\, dx \right\}^{1/p} \tag{5}$$

where M is independent of f, then

$$\left\{ \int_\alpha^\beta \left[\sum_{i=1}^{m} \varphi_i^2 \right]^{\frac{1}{2}p} d\xi \right\}^{1/p} \leq M \left\{ \int_a^b \left[\sum_{i=1}^{m} f_i^2 \right]^{\frac{1}{2}p} dx \right\}^{1/p} \tag{6}$$

for any system of functions $f_1, f_2, \cdots, f_m$ from S, and for $\varphi_i = T[f_i]$.[6]

The proof of (4) is similar to that of (6). Let us suppose for simplicity that $m = 2$, and let X_1 and X_2 be two independent random variables with Gaussian

[5] A transformation $\varphi = T[f]$ is said to be linear if

$$T[c_1 f_1 + c_2 f_2] = c_1 T[f_1] + c_2 T[f_2],$$

for c_1 and c_2 constant.

[6] See the second paper quoted in footnote [1].

distribution and dispersion 1. In other words, the distribution function $F(u)$ of both X_1 and X_2 is equal to

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-\frac{1}{2}t^2}\, dt.$$

Then, for any fixed values of X_1 and X_2,

$$(7) \qquad \int_{\alpha}^{\beta} | X_1\, d\varphi_1(\xi) + X_2\, d\varphi_2(\xi) | \leq M \int_{a}^{b} | X_1\, df_1(x) + X_2\, df_2(x) |.$$

Both sides here represent random variables, and their mean values $\mathfrak{M}\{\ \}$ satisfy the inequality

$$(8) \qquad \mathfrak{M}\left\{\int_{\alpha}^{\beta} | X_1\, d\varphi_1(\xi) + X_2\, d\varphi_2(\xi) |\right\} \leq \mathfrak{M}\left\{\int_{a}^{b} | X_1\, df_1(x) + X_2\, df_2(x) |\right\}.$$

For any fixed values of X_1 and X_2, the integral on the left of (7) is equal to

$$(9) \qquad \lim \sum_{i=1}^{n} | X_1\{\varphi_1(\xi_i) - \varphi_1(\xi_{i-1})\} + X_2\{\varphi_2(\xi_i) - \varphi_2(\xi_{i-1})\} |$$

$$(\alpha = \xi_0 < \xi_1 < \cdots < \xi_n = \beta)$$

as $\operatorname{Max} (\xi_i - \xi_{i-1}) \to 0$.
The expression under the limit sign here does not exceed

$$(|X_1| + |X_2|)(\operatorname{Var} \varphi_1 + \operatorname{Var} \varphi_2),$$

where $\operatorname{Var} \varphi_i$ means the total variation of φ_i over the interval (α, β). It follows that the left hand side of (8) is equal to

$$(10) \qquad \lim \sum_{i=1}^{n} \mathfrak{M}\{X_1[\varphi_1(\xi_i) - \varphi_1(\xi_{i-1})] + X_2[\varphi_2(\xi_i) - \varphi_2(\xi_{i-1})]\}.$$

(The theorem we use here is analogous to the theorem of Lebesgue that any convergent sequence of functions majorized by an integrable function may be integrated term by term.) It may now be observed that, for any constants γ_1, γ_2, the distribution function of $\gamma_1 X_1 + \gamma_2 X_2$ is the same as the distribution function of $(\gamma_1^2 + \gamma_2^2)^{\frac{1}{2}} X_1$. Hence the left hand side of (10) is equal to

$$C \lim \sum_{i=1}^{n} \{[\varphi_1(\xi_i) - \varphi_1(\xi_{i-1})]^2 + [\varphi_2(\xi_i) - \varphi_2(\xi_{i-1})]^2\},$$

where C is the mean value of $|X_1|$. It follows that the left hand side of (8) is equal to

$$C \int_{\alpha}^{\beta} \left\{\sum_{i=1}^{m} d\varphi_i^2\right\}^{\frac{1}{2}} \qquad (m = 2).$$

A similar value may be obtained for the right hand side of (8). Dividing both sides by C we get (4). This completes the proof of the theorem.

We add a few remarks.

(a) The above argument was based on that fact that

$$\mathfrak{M}\left\{\int_a^\beta |\, X_1\, d\varphi_1(\xi) + \cdots + X_m\, d\varphi_m(\xi)\,|\right\} = C\int_a^\beta \{d\varphi_1^2 + \cdots + d\varphi_m^2\}^{\frac{1}{2}},$$

where $X_1, X_2, \cdots, X_m$ are independent Gaussian variables with dispersion 1, and C is the mean value of $|X_1|$.

From the formula (3) we see that, if $f(x) = \text{const.}$ belongs to S, then the corresponding function $\varphi = T[f]$ is also constant, and we have equality in (3). Let us assume that we have equality in (3) *only* if $f = \text{const.}$ (such is the case, for example, if $\varphi = T[f] = T_n[f]$ is the n-th Fejér sum of the Fourier series of f.) Then it is easy to see that we have equality in (4) only if all the functions $f_1, f_2, \cdots, f_m$ are constants (so that both sides of (4) are then equal to 0). For equality in (4) implies equality in (8). Assuming for simplicity that $m = 2$ and taking into account our hypothesis concerning equality in (3), we see that there must exist four numbers $\lambda, \mu, \lambda', \mu'$ such that

$$\lambda f_1(x) + \mu f_2(x) = \text{const.}$$
$$\lambda' f_1(x) + \mu' f_2(x) = \text{const.}$$

and that $\lambda\mu' - \lambda'\mu \neq 0$. It follows that $f_1(x) = \text{const.}, f_2(x) = \text{const.}$

(b) If instead of Gaussian variables $X_1, X_2, \cdots$ we use the exceptional variables of Cauchy-Lévy,[7] we obtain instead of (4) the inequality

$$\int_a^\beta \{|\, d\varphi_1\,|^r + |\, d\varphi_2\,|^r + \cdots |\, d\varphi_m\,|^r\}^{1/r} \leq M\int_a^b \{|\, df_1\,|^r + \cdots + |\, df_m\,|^r\}^{1/r},$$

valid for every $1 \leq r \leq 2$.

(c) The inequality (6) as a consequence of (5) admits of a simple interpretation in the case $m = 2$. For let us assume that our operation $\varphi = T[f]$ is defined for real functions forming a linear family S, and that M in (5) is the least number satisfying that inequality. That is to say, M is the *norm* of the operation $\varphi = T[f]$. Let us now define the operation for all complex functions

$$f = f_1 + if_2, \qquad \text{with} \qquad f_1 \in S,\ f_2 \in S,$$

by the convention

$$T[f] = T[f_1] + iT[f_2]. \tag{11}$$

Then the inequality (6), applied to f_1 and f_2, may be written

$$\left\{\int_a^\beta |\, T[f]\,|^p\, d\xi\right\}^{1/p} \leq M\left\{\int_a^b |\, f\,|^p\, dx\right\}^{1/p}.$$

In other words, *the norm of the operation $\varphi = T[f]$ is not increased if the operation is extended to complex functions by the convention* (11).[8]

[7] See e.g., P. Lévy, *Calcul des probabilités*, 1925, p. 252. Cf. also the second paper of footnote [1].

[8] The general inequality (6), with m finite or infinite, may be given a similar interpretation if we consider classes $L^p\{l^r\}$ introduced by Bochner (see his "*Integration von Funk-*

(d) The inequality (6) may be extended to transformations carrying over L^p into L^q.[9] For some applications, however, the following generalization of (6) may be useful. Let $l(u)$ and $\lambda(u)$ be two non negative and non decreasing functions of $u (u \geq 0)$, which satisfy the inequalities

$$l(2u) \leq kl(u), \qquad \lambda(2u) \leq k\lambda(u),$$

with k independent of u. Let S be a linear family of functions $f(x)$ $(a \leq x \leq b)$ such that $l(|f|)$ is integrable over (a, b) for $f \in S$, and let $\varphi = T[f]$ be a linear transformation such that

$$\int_a^\beta \lambda(|\varphi|)\, d\xi \leq \int_a^b l(|f|)\, dx$$

for $f \in S$. Then for every system of functions f_i from S, we have the inequality

$$\int_a^\beta \Lambda\left(\left\{\sum_i \varphi_i^2\right\}^{\frac{1}{2}}\right) d\xi \leq \int_a^b L\left(\left\{\sum_i f_i^2\right\}^{\frac{1}{2}}\right) dx \tag{12}$$

where $\varphi_i = T[f_i]$ and the functions $L(u)$ and $\Lambda(u)$ are defined for $u \geq 0$ by the formulas

$$\begin{aligned} L(u) &= 2(2\pi)^{-\frac{1}{2}} u^{-1} \int_0^\infty l(v) \exp(-v^2/2u^2)\, dv, \\ \Lambda(u) &= 2(2\pi)^{-\frac{1}{2}} u^{-1} \int_0^\infty \lambda(v) \exp(-v^2/2u^2)\, dv. \end{aligned} \tag{13}$$

The proof of (12) does not differ essentially from that of (6).

§2. *On the best constant in a generalization of Bernstein's inequality*

Let $T(\theta) = T_n(\theta)$ be a trigonometric polynomial of order n, and let $\overline{T}(\theta)$ be the polynomial conjugate to $T(\theta)$. Generalizing the well known Bernstein inequality for $T'(\theta)$, Szegö showed[10] that

$$\operatorname*{Max}_\theta |T'(\theta)\cos\alpha + \overline{T}'(\theta)\sin\alpha| \leq n \operatorname*{Max}_\theta |T(\theta)|,$$

for any real α, the sign of equality being possible only in the case

$$T(\theta) = \sin(n\theta + \theta_0). \tag{1}$$

This fact may be interpreted in the following way: for any ordinary polynomial

$$P(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n$$

of degree n,

$$\operatorname*{Max}_{|z|=1} |P'(z)| \leq n \operatorname*{Max}_{|z|=1} |\Re P(z)|. \tag{2}$$

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tionen deren Werte die Elemente eines Vektorraumes sind," Fundamenta Math., 20 (1933), 262-276). The same interpretation may be given to an inequality slightly more general than (6) (see Theorem 2 of the second paper cited in footnote [3]).

[9] See Marcinkiewicz and Zygmund, *loc. cit.*

[10] Über einen Satz von Serge Bernstein, *Schriften der Königsberger Gelehrten Gesellschaft*, 1928.

There are other generalizations of Bernstein's inequality. For example, it has been shown[11] that

$$\left\{\int_0^{2\pi} |T'(\theta)|^p\,d\theta\right\}^{1/p} \leq n\left\{\int_0^{2\pi} |T(\theta)|^p\,d\theta\right\}^{1/p}, \tag{3}$$

$$\left\{\int_0^{2\pi} |\overline{T}'(\theta)|^p\,d\theta\right\}^{1/p} \leq n\left\{\int_0^{2\pi} |T(\theta)|^p\,d\theta\right\}^{1/p}, \tag{4}$$

for any $p \geq 1$, the sign of equality being again possible only in the case (1). Tamarkin[12] observed that the proof which gives (3) and (4) also gives the slightly more general inequality

$$\left\{\int_0^{2\pi} |T'(\theta)\cos\alpha + \overline{T}'(\theta)\sin\alpha|^p\,d\theta\right\}^{1/p} \leq n\left\{\int_0^{2\pi} |T(\theta)|^p\,d\theta\right\}^{1/p}, \tag{5}$$

where $p \geq 1$ and α is any real number. Equality here is again possible if and only if $T(\theta)$ is given by (1). This raises the question of the best possible constant C_p in the inequality

$$\left\{\int_0^{2\pi} |P'(e^{i\theta})|^p\,d\theta\right\}^{1/p} \leq C_p n\left\{\int_0^{2\pi} |\Re P(e^{i\theta})|^p\,d\theta\right\}^{1/p} \tag{6}$$

where $P(z)$ is any polynomial of degree n in z. From (3) and (4), and using Minkowski's inequality, we infer that $C_p \leq 2$, but 2 is certainly too big a value for C_p. For, if (2) is to follow from (6) for $p \to \infty$, we must have $C_p \to 1$. As a matter of fact, C_p is of the order $p^{\frac{1}{2}}$. More precisely, we shall prove the following

THEOREM. *For any polynomial*

$$P(z) = c_0 + c_1 z + \cdots + c_n z^n \tag{7}$$

of degree n, we have the inequality (6) *with*

$$C_p = \sqrt{\pi}\,\frac{\Gamma(\frac{1}{2}p + 1)}{\Gamma(\frac{1}{2}p + \frac{1}{2})}.$$

With this value of C_p, equality in (6) *occurs if and only if*

$$P(z) = Az^n, \tag{8}$$

where A is an arbitrary constant.

For the proof of this theorem we shall use an argument similar to that of §1.

Let X_1 and X_2 be two independent random variables with Gaussian distribution, and dispersion 1. Let $T(\theta)$ and $\overline{T}(\theta)$ denote respectively the real and the imaginary part of $P(e^{i\theta})$. For any fixed values of X_1 and X_2, the inequality (5) gives

$$\int_0^{2\pi} |X_1 T'(\theta) + X_2 \overline{T}'(\theta)|^p\,d\theta \leq n^p (X_1^2 + X_2^2)^{p/2} \int_0^{2\pi} |T(\theta)|^p\,d\theta. \tag{9}$$

[11] A. Zygmund, A remark on conjugate series, *Proc. London Math. Soc.*, 34 (1932), 392–400.

[12] J. D. Tamarkin, On the theory of polynomials of approximation, (mimeographed), Providence 1936.

Both sides here are random variables and their mean values satisfy the inequality

$$(10)\quad \mathfrak{M}\left\{\int_0^{2\pi} |\, X_1 T'(\theta) + X_2 \overline{T}'(\theta)\,|^p\, d\theta\right\} \leq n^p\, \mathfrak{M}\{(X_1^2 + X_2^2)^{p/2}\} \int_0^{2\pi} |\, T(\theta)\,|^p\, d\theta.$$

On both sides, the operations of integration and of taking mean values may be interchanged. Hence the right hand side of (10) is equal to

$$(11)\quad n^p \int_0^{2\pi} |\, T(\theta)\,|^p\, d\theta \cdot \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (X_1^2 + X_2^2)^{p/2} \exp\left[-\tfrac{1}{2}(X_1^2 + X_2^2)\right] dX_1 dX_2$$

or, passing to polar coordinates, equal to

$$(12)\quad 2^{p/2}\,\Gamma(\tfrac{1}{2}p + 1)\, n^p \int_0^{2\pi} |\, T(\theta)\,|^p\, d\theta.$$

Using the fact that for any constants γ_1 and γ_2 the distribution function of the variable $\gamma_1 X_1 + \gamma_2 X_2$ is the same as that of $(\gamma_1^2 + \gamma_2^2)^{\frac{1}{2}} X_1$, we see that the left hand side of (10) is equal to

$$\int_0^{2\pi} \{T'^2(\theta) + \overline{T}'^2(\theta)\}^{1/2}\, d\theta \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} |\, u\,|^p \exp(-\tfrac{1}{2}u^2)\, du$$

$$= 2^{p/2}\,\pi^{-1/2}\,\Gamma(\tfrac{1}{2}p + \tfrac{1}{2}) \int_0^{2\pi} \{T'^2(\theta) + \overline{T}'^2(\theta)\}^{1/2} d\theta.$$

In other words, the left hand side of (10) is equal to

$$(13)\quad \pi^{-1/2}\, 2^{p/2}\,\Gamma(\tfrac{1}{2}p + \tfrac{1}{2}) \int_0^{2\pi} |\, P'(e^{i\theta})\,|^p\, d\theta.$$

Comparing (12) and (13) we get the first part of the theorem.

That we have equality in (6) if $P(z)$ satisfies (8), follows from the fact that we then have equality in (9) for all values of X_1 and X_2, and that all subsequent operations preserve the sign of equality. (This could also be verified by straightforward computation, if Gauss' formula for $\Gamma(2u)$ is applied.) Conversely, the same argument shows that if we have equality in (6) we must have equality in (9) for at least one pair of numbers X_1, X_2. Hence, $T(\theta)$ must be of the form (1), and so $P(z)$ must satisfy (8). This completes the proof of the theorem.

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SUR LES FONCTIONS ANALYTIQUES DE DEUX VARIABLES COMPLEXES*

PAR S. BERGMAN ET J. MARCINKIEWICZ

§1. Dans la théorie des fonctions analytiques d'une variable complexe, on peut se borner souvent à l'étude des fonctions analytiques dans le cercle-unité. En effet, la fonction $f(z)$ étant régulière dans n'importe quel domaine simplement connexe, on peut le ramener à un cercle par une transformation conforme.

Il n'y a rien de semblable dans la théorie des fonctions analytiques de deux variables complexes, de sorte que non seulement les difficultés, mais aussi les résultats mêmes des recherches dépendent d'une manière extrêmement étroite du domaine dans lequel ces fonctions sont considérées. On peut donc ou bien envisager les classes les plus vastes de ces domaines et chercher à y établir les théorèmes les plus généraux, donc faibles, ou bien partir des domaines d'une structure spéciale, notamment telle qu'il soit possible de reproduire pour eux quelques procédés de la théorie des fonctions d'une variable complexe et, utilisant ces procédés, établir quelques théorèmes aussi forts que possible.

La deuxième méthode nous semble utile dans différentes recherches concernant le comportement des fonctions analytiques de deux variables complexes à la frontière du domaine de leur existence, p. ex. lorsqu'il s'agit de généraliser les théorèmes du type de Fatou. Pour étudier ces questions, nous considérons le cas des *domaines possédant une surface remarquable*[1] c. à d. tels qu'à leur frontière (à 3 dimensions) se trouve située une surface (à 2 dimensions), qui joue un rôle analogue à celui de la courbe frontière des domaines plans dans la théorie des fonctions d'une variable complexe.

La méthode utilisée est basée:

1^0 sur la possibilité de présenter chaque domaine envisagé comme l'ensemble-somme de domaines qui sont des images pseudo-conformes de bicylindres et dont les surface remarquables sont situées à la frontière du domaine considéré,[2]

2^0 sur l'application des procédés des évaluations maximales de la théorie des fonctions analytiques d'une variable complexe à l'étude des fonctions dans les domaines qui sont des images pseudo-conformes d'un bicylindre.

Pour mieux faire ressortir les principes de cette méthode, nous nous bornerons à une classe assez spéciale de domaines d'existence, à savoir aux domaines ayant une surface remarquable.

Commençons par quelques notions auxiliaires.

Nous appelons un domaine D de deux variables complexes *domaine de type* **R**, ou plus brièvement *domaine* **R**, si les conditions suivantes sont vérifiées:

* The above paper is an exact copy of one originally scheduled to appear in Fundamenta Mathematicae, vol. 33, p. 75–94, 1939, and abstracted in Mathematical Reviews, vol. 1, p. 10, 1940, from reprints. Presumably the volume 33 of Fundamenta will not appear in the immediate future.

[1] Bergman [2].

[2] Bergman [1].

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(A) La frontière du domaine D (cf. [2], figs. 10a–10c) consiste de deux hypersurfaces:

$$[z_1 = h(z_2, \lambda) \qquad (0 \leqslant \lambda \leqslant 2\pi), \qquad |z_2| \leqslant 1,]$$

$$\text{et } [z_1 = sh(z_2, \lambda) \qquad (0 \leqslant \lambda \leqslant 2\pi, s \leqslant 1), \qquad z_2 = e^{i\varphi_2}]$$

où $h(z_2, 0) = h(z_2, 2\pi)$ et, pour tout λ fixé, $h(z_2, \lambda)$ est une fonction analytique de z_2 et satisfait aux inégalités:

$$\min_{z_2, \lambda} |h(z_2, \lambda)| > \Delta > 0, \tag{1.1}$$

$$1/M \leqslant |h'_\lambda(z_2, \lambda)| \leqslant M < \infty, \qquad h'_\lambda = dh/d\lambda. \tag{1.2}$$

(B) Etant donnée l'équation

$$\rho = \rho(\omega) = \rho(z_2, \omega) \tag{1.3}$$

en coordonnées polaires de la courbe $z_1 = h(z_2, \lambda)$ $(0 \leqslant \lambda \leqslant 2\pi$, z_2 fixé), ρ est uniforme et on a:

$$|d\rho/d\omega| \leqslant M, \qquad |d^2\rho/d\omega^2| \leqslant M, \tag{1.4}$$

où M désigne une constante absolue et ω une fonction de λ telle que

$$1/M \leqslant |d\omega/d\lambda| \leqslant M. \tag{1.5}$$

Remarquons que la surface

$$z_1 = h(z_2, \lambda), \qquad z_2 = e^{i\varphi_2} \qquad (0 \leqslant \lambda \leqslant 2\pi, 0 \leqslant \varphi_2 \leqslant 2\pi)$$

forme la surface remarquable du domaine D.

Etant donnée une fonction continue $f(\lambda, \varphi_2)$ de deux variables λ et φ_2, nous désignons:

par $F(z_1, \varphi_2)$ la fonction harmonique dans le domaine borné par la courbe $z_1 = h(e^{i\varphi_2}, \lambda)$ $(0 \leqslant \lambda \leqslant 2\pi)$ et telle que $F(z_1, \varphi_2) = f(\lambda, \varphi_2)$ pour $z_1 = h(e^{i\varphi_2}, \lambda)$,

par $U(\varphi_1, \varphi_2)$ la valeur de la fonction F au point $z_1 = \Delta e^{i\varphi_1}$.

Les parties réelle et imaginaire seront désignées respectivement par Re et Im.

Nous dirons que la fonction continue et périodique $f(\lambda, \varphi_2)$ vérifie la condition[3] **N** si l'on a:

$$\int_0^{2\pi}\int_0^{2\pi} U(\varphi_1, \varphi_2) \cos(m\varphi_1 - n\varphi_2)\, d\varphi_1\, d\varphi_2 = 0$$

$$\int_0^{2\pi}\int_0^{2\pi} U(\varphi_1, \varphi_2) \sin(m\varphi_1 - n\varphi_2)\, d\varphi_1\, d\varphi_2 = 0 \qquad (m, n = 1, 2, \cdots). \tag{1.6}$$

La fonction $f(\lambda, \varphi_2)$ sera appelée *de classe* L^p $(p > 1)$ si l'on a

$$\int_0^{2\pi}\int_0^{2\pi} |f(\lambda, \varphi_2)|^p\, d\lambda\, d\varphi_2 < \infty; \tag{1.7}$$

[3] Bergman [1], en particulier p. 613.

nous dirons qu'elle vérifie la *condition* **N** si, pour tout $\epsilon > 0$, il existe une fonction continue et périodique $g(\lambda, \varphi_2)$ satisfaisant à cette condition et à la condition suivante:

$$\int_0^{2\pi}\int_0^{2\pi} |f(\lambda, \varphi_2) - g(\lambda, \varphi_2)|^p \, d\lambda \, d\varphi_2 \leqslant \epsilon. \tag{1.8}$$

Nous dirons qu'une fonction analytique $F(z_1, z_2)$ régulière dans un domaine D de type **R** apparatient à la *classe* H_p $(p \geqslant 1)$ si l'on a

$$\int_0^{2\pi}\int_0^{2\pi} |F(s_1 h(s_2 e^{i\varphi_2}, \lambda), s_2 e^{i\varphi_2})|^p \, d\lambda \, d\varphi_2 \leqslant M < \infty \tag{1.9}$$

dès que le domaine $D_{s_1 s_2}$ dont la frontière est

$$\begin{gathered} [z_1 = s_1 h(s_2 z_2, \lambda), \qquad |z_2| \leqslant 1] \\ \text{et } [|z_1| = s|h(s_2 e^{i\varphi_2}, \lambda)|, \qquad \arg z_1 = \arg h(s_2 e^{i\varphi_2}, \lambda), \\ (0 \leqslant s \leqslant s_1) \qquad z_2 = s_2 e^{i\varphi_2}] \end{gathered} \tag{1.10}$$

est contenu dans D.

Enfin, nous désignerons par $E(\lambda, \varphi_2, \alpha)$ la partie commune de D et du domaine défini par les inégalités:

$$|\arg(z_2 + e^{i\varphi_2}) - \varphi_2| \leqslant \alpha, \quad |\arg(z_1 - h(z_2, \lambda)) - \beta(z_2, \lambda)| \leqslant \alpha, \tag{1.11}$$

où $\beta(z_2, \lambda)$ désigne la direction de la normale extérieure à la courbe $z_1 = h(z_2, \lambda)$ $(0 \leqslant \lambda \leqslant 2\pi)$ au point $z_1 = h(z_2, \lambda)$.

Le résultat principal du notre travail est le suivant

THÉORÈME 1. *Soient D un domaine* **R** *et $F(z_1, z_2)$ une fonction de la classe H_p (où $p > 1$). Pour presque tout point (λ, φ_2), la limite*

$$\begin{gathered} f(\lambda, \varphi_2) = \lim F(z_1, z_2) \\ \textit{pour} \quad (z_1, z_2) \to (h(e^{i\varphi_2}, \lambda), e^{i\varphi_2}) \quad \textit{et} \quad (z_1, z_2) \in E(\lambda, \varphi_2, \alpha) \quad (\alpha < \pi/2) \end{gathered} \tag{1.12}$$

existe et on a

$$\int_0^{2\pi}\int_0^{2\pi} |f(\lambda, \varphi_2)|^p \, d\lambda \, d\varphi_2 < \infty. \tag{1.13}$$

On a aussi le théorème suivant que nous allons établir d'abord:

THÉORÈME 2. *Soit $f(\lambda, \varphi_2)$ une fonction de classe L^p (où $p > 1$) vérifiant la condition* **N**. *Il existe alors une fonction $F(z_1, z_2)$ de classe H_p satisfaisant pour presque tout point (λ, φ_2) à la condition*

$$\begin{gathered} \lim \mathrm{Re}[F(z_1, z_2)] = f(\lambda, \varphi_2) \\ \textit{pour} \quad (z_1, z_2) \to (h(e^{i\varphi_2}, \lambda), e^{i\varphi_2}) \quad \textit{et} \quad (z_1, z_2) \in E(\lambda, \varphi_2, \alpha) \quad (\alpha < \pi/2). \end{gathered} \tag{1.14}$$

Le cas le plus intéressant du th. 1 est celui où la fonction $F(z_1, z_2)$ est bornée dans tout le domaine D. Ce cas présente une généralisation directe du théorème de Fatou.

§2. LEMME 1. *Soit $f(\varphi)$ une fonction, de période 2π, définie dans l'intervalle $\langle 0, 2\pi\rangle$ et satisfaisant à la condition*

$$\int_0^{2\pi} |f(\varphi)|^p \, d\varphi < \infty \qquad (p > 1). \tag{2.1}$$

Alors, en posant

$$f^*(\varphi) = \max_h \frac{1}{h} \int_{-h}^{+h} |f(\varphi + u)| \, du, \tag{2.2}$$

on a

$$\int_0^{2\pi} f^{*p}(\varphi) \, d\varphi \leqslant A_p \int_0^{2\pi} |f(\varphi)|^p \, d\varphi, \tag{2.3}$$

où A_p ne dépend que de p.

Ce lemme est connu.[4]

LEMME 2. *La fonction $f(\varphi)$ vérifiant les hypotheses du lemme 1, posons:*

$$f(\varphi) \sim \frac{a_0}{2} + \sum^{\infty} (a_n \cos n\varphi + b_n \sin n\varphi), \tag{2.4}$$

$$U(r, \varphi) = \frac{a_0}{2} + \sum_1^{\infty} r^n (a_n \cos n\varphi + b_n \sin n\varphi), \tag{2.5}$$

$$U^*(\varphi) = \max_{|\omega - \varphi| \leqslant k(1-r) < \pi/2} [U(r, \omega - \varphi)]. \tag{2.6}$$

On a alors

$$U^*(\varphi) \leqslant A f^*(\varphi) \qquad (A = A(k)). \tag{2.7}$$

Ce lemme est aussi connu[5] et se démontre comme il suit. Nous pouvons toujours admettre que $\varphi = 0$. On a

$$U(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\omega) \, \frac{(1 - r^2) \, d\omega}{(1 - r)^2 + 4r \sin^2 \dfrac{\theta - \omega}{2}},$$

ce qui donne

$$|U(r, \theta)| \leqslant \frac{A}{2\pi} \int_{-\pi}^{\pi} |f(\omega)| \, \frac{(1 - r) \, d\omega}{(1 - r)^2 + \omega^2} = \frac{A}{2\pi} \int_{-\pi}^{0} \cdots + \frac{A}{2\pi} \int_0^{\pi} \cdots = B_1 + B_2.$$

En intégrant par parties, on trouve

$$B_2 = \frac{A}{2\pi} F(\omega) \, \frac{1 - r}{(1 - r)^2 + \omega^2} \Big|_0^{\pi} + \frac{A}{2\pi} \int_0^{\pi} F(\omega) \, \frac{2(1 - r)\omega \, d\omega}{[(1 - r)^2 + \omega^2]^2} \tag{2.8}$$

où

$$F(\omega) = \int_0^{\omega} |f(\varphi)| \, d\varphi.$$

[4] Hardy et Littlewood [4].

[5] Ibidem.

Or, $F(\omega) \leqslant \omega f^*(0)$, d'où selon (2.8)

$$B_2 \leqslant Af^*(0) + Af^*(0) \int_0^r \frac{(1-r)\omega^2\, d\omega}{[(1-r)^2+\omega^2]^2} \leqslant 3Af^*(0). \tag{2.9}$$

Un raisonnement analogue donne $B_1 \leqslant 3Af^*(0)$, ce qui démontre l'inégalité (2.7).

§3. Lemme 3. *Soit G un domaine borné par la courbe Γ dont l'équation en coordonnées polaires est*

$$\rho = \rho(\omega) \qquad (0 \leqslant \omega \leqslant 2\pi). \tag{3.1}$$

Supposons qu'on ait:

$$|\rho'(\omega)| \leqslant K, \qquad |\rho''(\omega)| \leqslant K \qquad (0 \leqslant \omega \leqslant 2\pi), \tag{3.2}$$

$$\min_{0 \leqslant \omega \leqslant 2\pi} \rho(\omega) = \lambda > 0. \tag{3.3}$$

Il existe alors un $r = r(\lambda, K)$ tel que tout cercle de rayon r, tangent à la courbe Γ et dont le centre se trouve dans G, est entièrement contenu dans G.

Considérons p. ex. le point $\omega = 0$. D'après (3.2), on a

$$\rho = \rho(0) + a\omega + b\omega^2, \tag{3.4}$$

où $a = \text{const.}$, $b = b(\omega)$, $|a| \leqslant K$ et $|b| \leqslant K$. Il en résulte que la courbe Γ_1 dont l'équation est

$$\rho = \rho(0) + a\omega - K\omega^2 \tag{3.5}$$

se trouve dans G. Désignons par P le point $\omega = 0$, $\rho = \rho(0)$ et par K_r la circonférence tangente au point P à la courbe Γ_1 et dont le centre C est situé sur la normale intérieure de la courbe Γ_1 au point P. Soit r le rayon de K_r. Désignons le point $\rho = 0$ par O et posons $x = \sphericalangle POC$. L'équation de K_r est évidemment:

$$\tilde{\rho}^2 + d^2 - 2\tilde{\rho}d\cos(\omega - x) = r^2, \qquad (\tilde{\rho}(0) = \rho(0)), \tag{3.6}$$

où d désigne la longueur du segment OC, et $\tilde{\rho} = \tilde{\rho}(\omega)$. L'équation (3.6) donne:

$$\tilde{\rho}'(0) = \frac{\tilde{\rho}(0)d\sin x}{\tilde{\rho}(0) - d\cos x} = a, \tag{3.7}$$

$$\tilde{\rho}''(\omega) = -\frac{\tilde{\rho}d\cos(\omega - x)}{\tilde{\rho} - d\cos(\omega - x)} - \frac{\tilde{\rho}'d^2\sin(\omega - x)\cos(\omega - x)}{\tilde{\rho} - d\cos(\omega - x)} + \frac{\tilde{\rho}d^2\sin^2(\omega - x)}{[\tilde{\rho} - d\cos(\omega - x)]^2}. \tag{3.8}$$

L'expression $\tilde{\rho}(0) - d\cos x$ désigne la longueur de la projection du segment CP sur l'axe $\omega = 0$. Elle est donc égale à

$$r\sqrt{\frac{\tilde{\rho}^2(0)}{\tilde{\rho}^2(0) + a^2}} = r\alpha \quad \text{où} \quad \alpha = \cos \sphericalangle OPC.$$

D'autre part, pour $\eta = \eta(K, \lambda)$ suffisamment petit et pour $|\omega| < r\eta$, on a $|\tilde{\rho} - \tilde{\rho}(0)| \leqslant \frac{1}{3}r\alpha$, $|d \cos(\omega - x) - d \cos x| \leqslant \frac{1}{3}r\alpha$ et comme on a aussi $|x| \leqslant \pi r/\lambda$, on trouve facilement pour $|\omega| < \eta r$

$$\tilde{\rho}''(\omega) = -\frac{\tilde{\rho}(0)\sqrt{\tilde{\rho}^2(0) + a^2}}{r} + A,$$

où A est uniformément borné pour $|\omega| \leqslant \eta r$, $|a| \leqslant K$, $\rho(0) \geqslant \lambda$ et $r < \lambda/2$. Il vient pour l'équation de K_r

$$\tilde{\rho}(\omega) = \tilde{\rho}(0) + a\omega - \tfrac{1}{2}\omega^2\left[\frac{\tilde{\rho}(0)\sqrt{\tilde{\rho}^2(0) + a^2}}{2r} + A\right].$$

En choisissant r suffisamment petit, on voit qu'il existe un secteur de K_r qui se trouve entièrement dans G. Pour achever la démonstration, il suffit d'inscrire dans ce secteur le cercle tangent à Γ_1 au point P.

LEMME 4. *G et Γ vérifiant les hypothesès* (3.1)–(3.3), *il existe un $r = r(\lambda, K)$ tel que tout cercle de rayon r, tangent à Γ et dont le centre est situé sur la normale extérieure, se trouve situé en dehors de G.*

La démonstration est analogue à celle du lemme 3.

LEMME 5. *G et Γ vérifiant* (3.1) *où*

$$|\rho'(\omega)| \leqslant K, \tag{3.9}$$

$$\min_{0 \leqslant \omega \leqslant 2\pi} \rho \geqslant \lambda > 0, \tag{3.10}$$

soit $\Phi(z)$, $\Phi(0) = 0$, $0 \in G$, la fonction effectuant une représentation conforme du domaine G sur le cercle unité. Alors, pour un nombre quelconque $\Delta > 0$, il existe un nombre $\delta > 0$ ne dépendant que de Δ, K et λ, et tel que

$$\min_{z \in \Gamma} |z_0 - z| \geqslant \Delta \quad \textit{entraîne} \quad \min_{z \in \Gamma} |\Phi(z) - \Phi(z_0)| \geqslant \delta. \tag{3.11}$$

En effet, dans le cas contraire, il existerait:

1^0 une suite infinie de domaines G_n, bornés par les courbes Γ_n aux équations $\rho = \rho_n(\omega)$ assujetties à (3.9) et (3.10), les courbes Γ_n convergeant uniformément vers la courbe Γ[6] de manière que les fonctions $\Phi_n(z)$, correspondantes à Γ_n, convergent uniformément dans tout domaine intérieur à G.

2^0 une suite de points z_n tels que:

$$\min_{z \in \Gamma_n} |z_n - z| \geqslant \Delta, \qquad \min_{z \in \Gamma_n} |\Phi_n(z_n) - \Phi_n(z)| \leqslant 1/n, \qquad \lim_{n \to \infty} z_n = z_0.$$

Or, il est facile de voir que l'on aurait alors $\Phi(z_0) = \lim_{n \to \infty} \Phi_n(z_n)$ et $|\Phi(z_0)| = 1$, ce qui est impossible, la fonction $\Phi(z)$ donnant par hypothèse la représentation conforme de G sur le cercle-unité.

LEMME 6. *G et Γ vérifiant* (3.1)–(3.3), *soit $\Phi(z)$ la fonction donnant une représentation conforme du domaine G sur le cercle-unité, de manière que $\Phi(0) = 0$.*

[6] C. à d. $\rho_n(\omega)$ convergeant uniformément vers $\tilde{\rho}(\omega)$.

Alors, la dérivée $\Phi'(z)$ est continue dans le domaine fermé $G + \Gamma$ et on a $0 < |\Phi'(z)| < \infty$ pour tout $z \in G + \Gamma$.

Ce lemme est connu.[7]

LEMME 7. *Dans les mêmes hypothèses on a*

$$1/M \leqslant |\Phi'(z)| \leqslant M \quad \text{pour tout} \quad z \in G + \Gamma, \tag{3.12}$$

où M est une constante qui ne dépend que de K et λ.

Ce lemme est connu.[8] Comme il est pour nous très important, nous en donnons ici la demonstration complète.

Remarquons d'abord que, en vertu du lemme 6 et les principes de minimum et de maximum, il suffit de démontrer (3.12) pour $z \in \Gamma$. Soit p. ex. z_1 le point $\omega = 0$, $\rho = \rho(0)$. D'après le lemme 3, il existe un nombre $r = r(\lambda, K)$ tel que le cercle K de rayon r, tangent à Γ au point z_1, se trouve entièrement dans G. Soit z_0 le centre de K. Désignons respectivement par $F(z)$ et $H(z)$ les fonctions représentant conformément les domaines G et K sur le cercle-unité de manière que l'on ait $F(z_0) = H(z_0) = 0$ et $F(z_1) = H(z_1) = 1$.

D'après le principe de maximum, on a pour $z \in K$

$$\lg |F(z)| \leqslant \lg |H(z)|,$$

ce qui donne évidemment

$$\frac{d}{dn} \lg |F(z_1)| \leqslant \frac{d}{dn} \lg |H(z_1)| = -\frac{1}{r},$$

où $\frac{d}{dn}$ désigne la dérivée prise dans la direction de la normale intérieure. On voit facilement que $\frac{d}{dn} \lg |F(z_1)| = F'(z_1)$, ce qui donne

$$|F'(z_1)| \geqslant 1/r. \tag{3.13}$$

La fonction

$$\Phi(z) = e^{i\theta} \frac{F(z) - F(0)}{1 - F(z)\overline{F(0)}}$$

donne la répresentation conforme du domaine G sur le cercle unité et l'on a $\Phi(0) = 0$. D'autre part, on peut choisir θ de manière que l'on ait $\Phi(z_1) = 1$. Il vient

$$\Phi'(z_1) = e^{i\theta} \frac{F'(z_1)[1 - |F(0)|^2]}{[1 - \overline{F(0)}]^2},$$

ce qui donne d'après (3.14) et lemme 5

$$|\Phi'(z_1)| \geqslant \frac{1}{4r}(1 - |F(0)|^2) = \frac{1}{4r}(1 - |\Phi(z_0)|^2) \geqslant \frac{\delta(r, \lambda, K)}{4r},$$

[7] Kellog [6], Seidel [9], Lichtenstein [7].

[8] Warschawski [10], [11] p. 433 et p. 440.

et la partie gauche de l'inégalité (3.12) se trouve établi. On en démontre la partie droit d'une manière tout à fait analogue, mais en se servant du lemme 4 au lieu du lemme 3.

LEMME 8. *Dans les hypothèses du lemme 6, posons:*

$$\arg \Phi(z) = \theta \quad \text{où} \quad z \in \Gamma, \qquad \arg z = \omega.$$

Désignons par D_θ le domaine contenu dans le cercle-unité et vérifiant l'inégalité

$$|\arg Z - \theta| \leqslant \kappa(1 - |Z|) < \pi/2,$$

par I le segment $y = x \operatorname{tg} \omega$, $\rho \leqslant \rho(\omega)$, où x et y désignent les coordonnées cartésiennes, et par C_θ l'image de I.

Il existe alors, pour un κ convenablement choisi, un $R = R(K, \lambda) < 1$ tel que la courbe C_θ est contenue dans $D_\theta + K_R$, où K_R désigne le cercle $|Z| \leqslant R$.

En tenant compte de (3.2) et (3.3), on conclut qu'il existe un nombre N ne dépendant que de K et λ, et tel que

$$|z_0 - \rho(\arg z_0) e^{i \arg z_0}| \leqslant N \min_{z_1 \in \Gamma} |z_0 - z_1|.$$

D'autre part, (3.12) donne pour tout couple z_1, z_2 de points de $G + \Gamma$

$$M|z_1 - z_2| > |\Phi(z_1) - \Phi(z_2)| > \frac{1}{M}|z_1 - z_2|,$$

d'où l'inégalité $|\Phi(z_1) - \Phi(z_0)| \geqslant \dfrac{1}{M^2 N}[1 - |\Phi(z_0)|]$ où $z_1 \in \Gamma$ et $z_0 \in I$.

La dernière formule entraîne sans difficulté le lemme.

LEMME 9. *G et Γ vérifiant les conditions (3.1)–(3.3), soient $f(\omega)$ une fonction de classe L^p où $p > 1$, et $F(z)$ une fonction harmonique dans G satisfaisant presque partout à la condition $F(\rho e^{i\omega}) = f(\omega)$.*

Alors, en posant

$$F^*(\omega) = \max_{0 \leqslant \rho \leqslant \rho(\omega)} |F(\rho e^{i\omega})|,$$

on a

$$\int_0^{2\pi} \{F^*(\omega)\}^p \, d\omega \leqslant A \int_0^{2\pi} |f(\omega)|^p \, d\omega$$

où la constante A ne dépend que de p, K et λ.

C'est une conséquence immédiate des lemmes 1, 2 et 8.

§4. La partie réelle $U(z_1, z_2)$ d'une fonction analytique $F(z_1, z_2)$ de deux variables complexes $z_1 = x_1 + iy_1$ et $z_2 = x_2 + iy_2$ satisfait aux équations:

$$\frac{\partial^2 U}{\partial x_1^2} + \frac{\partial^2 U}{\partial y_1^2} = 0, \tag{4.1}$$

$$\frac{\partial^2 U}{\partial x_2^2} + \frac{\partial^2 U}{\partial y_2^2} = 0, \tag{4.2}$$

$$\frac{\partial^2 U}{\partial x_1 \partial x_2} + \frac{\partial^2 U}{\partial y_1 \partial y_2} = 0, \tag{4.3}$$

$$\frac{\partial^2 U}{\partial x_1 \partial y_2} - \frac{\partial^2 U}{\partial x_2 \partial y_1} = 0. \tag{4.4}$$

Une fonction $U(z_1, z_2)$ qui satisfait aux équations (4.1) et (4.2) s'appelle *doublement harmonique*;[9] si elle satisfait aussi aux équations (4.3) et (4.4), on l'appelle *biharmonique*.

LEMME 10. *Soit $U(z_1, z_2)$ une fonction doublement harmonique dans un domaine D et biharmonique dans un sous-domaine D' de D tel que toute intersection $z_2 = \text{const.}$ contient un cercle situé dans le domaine D'. Alors, la fonction $U(z_1, z_2)$ est biharmonique dans le domaine D tout entier.*

Considérons la fonction

$$H(z_1, z_2) = \frac{\partial^2 U}{\partial x_1 \partial x_2} + \frac{\partial^2 U}{\partial y_1 \partial y_2}.$$

C'est une fonction harmonique de z_1 pour tout z_2 fixe. Or, comme elle s'annule dans un cercle du plan $z_2 = \text{const.}$, elle s'annule identiquement dans D.

L'équation (4.4) se démontre d'une manière analogue.

§5. $f(\lambda, \varphi_2)$ étant une fonction de classe L^p $(p > 1)$, soit $F(z_1, \varphi_2)$ une fonction harmonique dans le domaine G_{φ_2} borné par la courbe Γ_{φ_2}, $z_1 = h(e^{i\varphi_2}, \lambda)$ et qui admet $f(\lambda, \varphi_2)$ comme fonction frontière.[10] Soit $u(\varphi_1, \varphi_2)$ la valeur de la fonction $F(z_1, \varphi_2)$ au point $z_1 = \Delta e^{i\varphi_1}$, où Δ satisfait à (1.1). On conclut du lemme 9 que $u \in L^p$, ce qui permet de définir à l'aide de l'intégrale de Poisson une fonction $U(z_1, z_2)$ doublement harmonique dans le bicylindre $|z_2| < 1$, $|z_1| < \Delta$; désignons-le par B.

En supposant la condition **N** satisfaite, la fonction $U(z_1, z_2)$ est biharmonique. A plus forte raison elle est biharmonique dans le domaine D_{s_0} dont la frontière est:

$$[z_1 = s_0 h(z_2, \lambda), \quad |z_2| \leqslant 1] \quad \text{et} \quad [z_1 = s h(e^{i\varphi_2}, \lambda), \quad z_2 = e^{i\varphi_2}]$$

où $0 \leqslant \lambda \leqslant 2\pi$, $0 \leqslant s \leqslant s_0$ et s_0 est assez petit pour que $|s_0 h(z, \lambda)| < \Delta$. La famille des courbes $\Gamma_{z_2}(s_0)$, $z_1 = s_0 h(z_2, \lambda)$, $z_2 = \text{const.}$ vérifie les conditions (1.1) et (1.4). Il existe donc un r satisfaisant à la thèse du lemme 3. Fixons $\lambda = \lambda_0$ et s_1 tel que $s_0 < s_1 \leqslant 1$; considérons la transformation:[11]

$$z_1' = \frac{z_1 - s_1 h(z_2, \lambda_0)}{i h_\lambda'(z_2, \lambda_0)}, \qquad z_2' = z_2. \tag{5.1}$$

Par (5.1) le domaine D_{s_1} se transforme en un domaine D_{s_1}' et—comme on le vérifie facilement—cette transformation conserve les conditions (1.1) et (1.4), et même d'une façon uniforme par rapport aux constantes K et Δ. Désignons:

par $\Gamma_{z_2}(s_1)$ la courbe $z_1 = s_1 h(z_2, \lambda)$, $z_2 = \text{const.}$

par $C_{\lambda_0}(s_1)$ la courbe $z_1 = s_1 h(z_2, \lambda_0)$, $y_2 = \operatorname{Im} z_2 = \text{const.}$

par $\Gamma_{z_2}'(s_1)$ et $C_{\lambda_0}'(s_1)$ respectivement les courbes resultant

de $\Gamma_{z_2}(s_1)$ et $C_{\lambda_0}(s_1)$ par la transformation (5.1).

On voit aisément que $C_{\lambda_0}'(s_1)$ coïncide avec la droite $z_1' = 0$, $y_2' = \operatorname{Im} z_2 = \text{const.}$ et que la direction de la normale à la courbe $\Gamma_{z_2}(s_1)$ au point z_1' est celle de l'axe

[9] Bergman [1].

[10] Bergman [1].

[11] Bergman [3].

Im $z_1' = 0$. Il en résulte que le domaine D'_{s_1} contient un bicylindre $B'_r(\lambda_0)$ de la forme $|z_1' + r| \leqslant r$, $|z_2'| \leqslant 1$. (cf. [3]) Posons:

$$v(\psi_1, \psi_2) = U(z_1', \varphi_2), \qquad |z' + r| = r, \qquad \psi_1 = \arg(z_1' + r) \text{ et } \psi_2 = \varphi_2.$$

En tenant compte du lemme 9, on conclut facilement que $v \in L^p$, ce qui permet de définir dans $B'_r(\lambda_0)$ une fonction doublement harmonique $V(z_1', z_2')$ admettant v comme fonction frontière.

Si s_1 est suffisamment proche de s_0, le domaine $D'_{s_0} \cdot B'_r(\lambda_0)$ contient tout un bicylindre. Il en résulte que a fonction $V(z_1', z_2')$ ainsi définie dans le domaine $B'_r(\lambda_0)$ peut être considérée comme le prolongement de la fonction $U(z_1', z_2')$ sur $B'_r(\lambda_0)$. Le lemme 10 montre donc qu'elle est biharmonique dans le domaine $D'_{s_0} + B'_r(\lambda_0)$.

En effectuant la représentation inverse de (5.1), on obtient la fonction biharmonique définie dans le domaine $D_{s_0} + B_r(\lambda_0)$ où $B_r(\lambda_0)$ désigne le transformé de $B'_r(\lambda_0)$. Maintenant, on peut choisir une suite $\lambda_0, \lambda_1, \cdots, \lambda_n$ de manière que le domaine $D_{s_0} + B_{\lambda_0} + B_{\lambda_1} + \cdots + B_{\lambda_n}$ contienne un domaine $D_{s'}$ où $s' > s_0$. Cette méthode permet ainsi de définir la fonction $U(z_1, z_2)$ dans un domaine plus grand et on prouve facilement que $s' - s_0$ est de l'ordre de grandeur de r. On peut donc définir une suite infinie $D_{s_0}, D_{s_1}, D_{s_2}, \cdots$ de domaines croissants, de manière que D_{s_k} tende vers D et que l'on puisse passer d'un domaine à l'autre en répétant les opérations décrites.

LEMME 11. *La fonction $U(z_1, z_2)$ ainsi définie est de classe H_p dans D.*

C'est une conséquence immédiate du lemme 9 (appliqué deux fois).

§6. LEMME 12. *$f(\varphi_1, \varphi_2)$ étant une fonction de classe L^p où $p > 1$, posons*

$$(6.1) \qquad f^*(\varphi_1, \varphi_2) = \max_{hk} \frac{1}{hk} \int_{-h}^{+h} \int_{-k}^{+k} |f(\varphi_1 + u_1, \varphi_2 + u_2)| \, du_1 du_2.$$

On a alors

$$(6.2) \qquad \int_0^{2\pi} \int_0^{2\pi} \{f^*(\varphi_1, \varphi_2)\}^p \, d\varphi_1 d\varphi_2 \leqslant A_p \int_0^{2\pi} \int_0^{2\pi} |f(\varphi_1, \varphi_2)|^p \, d\varphi_1 d\varphi_2.$$

Ce lemme est connu.[12]

LEMME 13. *$f(\varphi_1, \varphi_2)$ étant une fonction de classe L^p où $p > 1$ et $U(z_1, z_2)$ une fonction doublement harmonique dans le bicylindre $|z_1| < 1$, $|z_2| < 1$, admettant $f(\varphi_1, \varphi_2)$ comme la fonction frontière, posons:*

$$U^*(\varphi_1, \varphi_2) = \max |U(z_1, z_2)| \quad \textit{pour} \quad |\arg z_i - \varphi_i| \leqslant k(1 - r_i) \leqslant \pi/2$$

où $i = 1, 2$ et k est une constante. On a alors

$$(6.3) \qquad \int_0^{2\pi} \int_0^{2\pi} \{U^*(\varphi_1, \varphi_2)\}^p \, d\varphi_1 d\varphi_2 \leqslant A_p \int_0^{2\pi} \int_0^{2\pi} |f(\varphi_1, \varphi_2)|^p d\varphi_1 d\varphi_2 \quad (p > 1).$$

Ce lemme est aussi connu[13] et se démontre de la même manière que le lemme 2.

LEMME 14. *Soit $\Gamma(\varphi_2)$ une famille de courbes dans le plan z aux equations*

[12] Jessen, Marcinkiewicz et Zygmund [5].

[13] Ibidem.

$z = h(\lambda, \varphi_2)$ où $0 \leqslant \lambda \leqslant 2\pi$, bornant les domaines $G(\varphi_2)$. Supposons qu'en les écrivant en coordonnées polaires

$$\rho = \rho(\varphi_1) = \rho(\varphi_1, \varphi_2) \quad où \quad \varphi_1 = \arg z,$$

on a

$$\left|\frac{d\rho}{d\varphi_1}\right| \leqslant K, \qquad \left|\frac{d^2\rho}{d\varphi_1^2}\right| \leqslant K, \qquad \min_{\varphi_1, \varphi_2} \rho \geqslant \Delta > 0$$

et que λ est une fonction croissante de φ_1 (avec φ_2 fixe) pour laquelle

$$\frac{1}{M} \leqslant \left|\frac{d\lambda}{d\varphi_1}\right| \leqslant M, \qquad \lambda(0) = 0. \tag{6.4}$$

Soient:

$U(z, \varphi_2)$ la fonction harmonique dans le domaine $G(\varphi_2)$ limité par la courbe $\Gamma(\varphi_2)$ et $f(\lambda, \varphi_2)$ sa fonction frontière;

K_r le cercle de rayon $r < \Delta/2$ satisfaisant à la thèse du lemme 3, situé dans le domaine $G(\varphi_2)$ et tangent à la courbe $\Gamma(\varphi_2)$ au point $z_1 = h(0, \varphi_2)$;

z_0 le centre de K_r,

$$\psi = \psi(z) = \arg(z - z_0) - \arg(z_1 - z_0),$$

$$V(\psi, \varphi_2) = U(z, \varphi_2), \qquad |z - z_0| = r, \qquad \psi = \psi(z).$$

Enfin, posons

$$U^*(\lambda, \varphi_2) = \operatorname*{borne\,sup}_{z \in I_\lambda} |U(z, \varphi_2)|$$

où I_λ désigne le segment joignant les points 0 et $z = h(\lambda, \varphi_2)$.

Alors on a

$$\operatorname*{borne\,sup}_{h,k} \frac{1}{hk} \int_h^{+h} \int_k^{+k} |V(\psi, \varphi_2 + \varphi)|\, d\psi\, d\varphi$$

$$\leqslant A \operatorname*{borne\,sup}_{h,k} \frac{1}{hk} \int_h^h \int_k^k U^*(\lambda, \varphi_2 + \varphi)\, d\lambda\, d\varphi$$

où A ne dépend que de K, M et Δ.

Il est facile de voir qu'il existe un nombre $\delta > 0$ tel que ψ soit, pour $|\psi| < \delta$, une fonction de φ_1 et φ_2 assujettie à la condition

$$\frac{1}{A_1} \leqslant \left|\frac{\partial\psi}{\partial\varphi_1}\right| \leqslant A_1 \tag{6.5}$$

où A_1 est une constante ne dépendant que de Δ, K, M_1 et δ. Or, on a

$$|V(\psi, \varphi_2)| \leqslant U_*(\varphi_1, \varphi_2) = \max_{\arg z = \varphi_1} |U(z, \varphi_2)|.$$

On en tire pour $h < \delta$

$$\frac{1}{hk} \int_h^h \int_k^k |V(\psi, \varphi_2 + \varphi)|\, d\psi\, d\varphi \leqslant \frac{1}{hk} \int_h^h \int_k^k |U_*(\varphi_1, \varphi_2 + \varphi)|\, d\psi\, d\varphi$$

$$\leqslant \frac{A_1}{hk} \int_{h'}^{h'} \int_k^k U_*(\varphi_1, \varphi_2 + \varphi)\, d\varphi_1\, d\varphi$$

où h' désigne la valeur de φ_1 lorsque $\psi = h$ pour φ_2 fixé. On conclut de (6.5) et (6.4) que

$$\frac{1}{hk}\int_{-h}^{h}\int_{-k}^{k} |V(\psi, \varphi_2 + \varphi)|\, d\psi\, d\varphi \leqslant \frac{M_1}{hk}\int_{-h}^{h}\int_{-k}^{k} U_*(\varphi_1, \varphi_2 + \varphi)\, d\varphi_1\, d\varphi$$

$$\leqslant B \text{ borne sup } \frac{1}{hk}\int_{-h}^{h}\int_{-k}^{k} U^*(\lambda, \varphi_2 + \varphi)\, d\lambda\, d\varphi.$$

D'autre part, si $h > \delta$ on a d'après le lemme 9

$$\frac{1}{hk}\int_{-h}^{h}\int_{-k}^{k} |V(\psi, \varphi_2 + \varphi)|\, d\psi\, d\varphi \leqslant \frac{1}{\delta k}\int_{\pi}^{\pi}\int_{-k}^{k} |V(\psi, \varphi_2 + \varphi)|\, d\psi\, d\varphi$$

$$\leqslant \frac{A}{\delta k}\int_{\pi}^{\pi}\int_{-k}^{k} |U_*(\varphi_1, \varphi_2 + \varphi)|\, d\varphi_1\, d\varphi$$

$$\leqslant B(K, M, \Delta, \delta) \limsup_{h,k} \frac{1}{hk}\int_{-h}^{h}\int_{-k}^{k} U^*(\lambda, \varphi_2 + \varphi)\, d\lambda\, d\varphi$$

ce qui achève la démonstration du lemme 14.

§7. Ces preliminaires achevés, soient $f(\lambda, \varphi_2)$ une fonction de classe L^p (où $p > 1$) satisfaisant à la condition **N** et $U(z_1, z_2)$ la fonction biharmonique définie par les opérations décrites dans les §5 en partant de la fonction $f(\lambda, \varphi_2)$. Fixons λ_0 et φ_2 et considérons la transformation

$$z_1' = \frac{z_1 - h(z_2, \lambda_0)}{ih_\lambda'(z_2, \lambda_0)}, \qquad z_2' = z_2. \tag{7.1}$$

D'après le lemme 3, le transformé D' du domaine D contient un bicylindre B de la forme $|z_1' + r| \leqslant r$, $|z_2'| \leqslant 1$. En tenant compte des lemmes 13 et 14, on en conclut que l'on a

$$\max_{(z_1', z_2') \in B'(\lambda_0, \varphi_2)} |U(z_1', z_2')| \leqslant AU^*(\lambda_0, \varphi_2) = A \max_{\substack{z = ih(e^{i\varphi_2}\lambda_0) \\ 0 \leqslant s \leqslant 1}} |U(z_1, e^{i\varphi_2})|$$

où $B'(\lambda_0, \varphi_2)$ est le domaine défini par les inégalités:

$$|\arg(z_1' + r)| \leqslant k(1 - |z_1' + r|) < \pi/2, \qquad |\arg z_2' - \varphi_2| \leqslant k(1 - |z_2'|) < \pi/2.$$

Soit $B(\lambda_0, \varphi_2)$ le domaine qui s'obtient de $B'(\lambda_0, \varphi_2)$ par la transformation inverse de (7.1). Il vient

$$V(\lambda_0, \varphi_2) = \max_{(z_1, z_2) \in B(\lambda_0, \varphi_2)} |U(z_1, z_2)| \leqslant AU^*(\lambda_0, \varphi_2), \tag{7.2}$$

ce qui donne d'après les lemmes 9 et 12

$$\int_0^{2\pi}\int_0^{2\pi} V^p(\lambda, \varphi_2)\, d\lambda\, d\varphi_2 \leqslant A\int_0^{2\pi}\int_0^{2\pi} |f(\lambda, \varphi_2)|^p\, d\lambda\, d\varphi_2. \tag{7.3}$$

Posons $f = f_1 + f_2$ où f_1 est une fonction continue et

$$\int_0^{2\pi}\int_0^{2\pi} |f_2(\lambda, \varphi_2)|^p\, d\lambda\, d\varphi_2 \leqslant \epsilon^p. \tag{7.4}$$

Considérons les fonctions biharmoniques correspondantes $U_1(z_1, z_2)$ et $U_2(z_1, z_2)$. En appliquant la transformation (7.1), on prouve facilement qu'on a partout et d'une façon uniforme

$$\lim_{\substack{z_1 \to \lambda(e^{i\varphi_2}\lambda) \\ z_2 \to e^{i\varphi_2}}} U_1(z_1, z_2) = f_1(\lambda, \varphi_2).$$

D'autre part, l'inégalité (7.2) donne mes $\underset{\lambda,\varphi_2}{\boldsymbol{E}} [V_2(\lambda, \varphi_2) > \sqrt{\epsilon}] \leqslant A\sqrt{\epsilon}$, d'où

$$\text{mes} \underset{\lambda,\varphi_2}{\boldsymbol{E}} \Big(\overline{\lim_{\substack{(z_1,z_2) \in B(\lambda,\varphi_2) \\ z_1 \to \lambda(e^{i\varphi_2}\lambda) \\ z_2 \to e^{i\varphi_2}}}} |U(z_1, z_2) - f(\lambda, \varphi_2)| > \sqrt{\epsilon} \Big) < A\sqrt{\epsilon},$$

ce qui équivaut à (1.14) pour $U(z_1, z_2)$. Pour achever la démonstration du th. 2, il suffit donc de montrer que la fonction conjuguée de $U(z_1, z_2)$ est de classe H_p, ce qui revient à montrer que la fonction conjuguée à $f(\lambda, \varphi_2)$ est de classe L^p. Or, c'est une conséquence du lemme 7 et du lemme connu[14] suivant:

LEMMA 15. *$F(z) = U(z) + iV(z)$ étant une fonction analytique dan le cercle unité, l'inégalité* $\int_0^{2\pi} |U(e^{i\varphi})|^p \, d\varphi < \infty$ *où* $p > 1$ *entraîne* $\int_0^{2\pi} |V(e^{i\varphi})|^p \, d\varphi \leqslant A_p \int_0^{2\pi} |U(e^{i\varphi})|^p \, d\varphi$ *où A_p ne dépend que de p.*

Le th. 2 se trouve ainsi démontré.

§8. Passons à la démonstration du th 1.

LEMME 16. *Soit $f(z)$ une fonction analytique définie dans le cercle unité et satisfaisant à la condition*

$$\int_0^{2\pi} |f(re^{i\varphi})|^p \, d\varphi \leqslant M \qquad (r < 1, p > 1).$$

Il existe alors une fonction $f(e^{i\varphi})$ telle que

$$\lim_{r \to 1} \int_0^{2\pi} |f(re^{i\varphi}) - f(e^{i\varphi})|^p \, d\varphi = 0.$$

Ce lemme est connu.[15]

LEMME 17. *G et I vérifiant les hypothèses* (3.1)–(3.3), *soit $f(z)$ une fonction régulière dans G et telle que*

$$\int_\Gamma |f(sz)|^p \, |dz| \leqslant M \qquad (s < 1).$$

Il existe alors une fonction $f(z)$ définie pour $z \in \Gamma$ et telle que

$$\lim_{s \to 0} \int_\Gamma |f(z) - f(sz)|^p \, |dz| = 0.$$

C'est une conséquence des lemmes 7, 9 et 16.

[14] M. Riesz [8].

[15] Voir p. ex. Zygmund [12], surtout p. 86 et 87.

LEMME 18. *Soit $\Gamma(\varphi)$ une famille de courbes vérifiant uniformément les hypothèses* (3.1)–(3.3). *Admettons que, pour tout φ, il existe une fonction $F(z, \varphi)$ régulière dans le domaine $G(\varphi)$ borné par la courbe $\Gamma(\varphi)$ et que l'on ait*

$$\int_0^{2\pi} \int_{\Gamma(\varphi)} |F(sz, \varphi)|^p \, |dz| \, d\varphi \leqslant M \qquad (s < 1, p > 1).$$

Il existe alors une fonction $F(z, \varphi)$ définie pour $z \in \Gamma(\varphi)$ et pour laquelle on a

$$\lim_{s \to 0} \int_0^{2\pi} \int_{\Gamma(\varphi)} |F(sz, \varphi) - F(z, \varphi)|^p \, |dz| \, d\varphi = 0.$$

Posons

$$H(\varphi) = \text{borne sup}_{0 \leqslant s < 1} \int_{\Gamma(\varphi)} |F(sz, \varphi)|^p \, |dz|.$$

Nous allons établir l'inégalité

$$\int_0^{2\pi} H(\varphi) \, d\varphi < AM \tag{8.1}$$

où $A = A(K, \lambda, p)$.

Supposons d'abord que $H(\varphi) = \infty$ pour tout φ appartenant à un ensemble E de mesure positive. Le nombre K_1 étant fixé, on peut trouver pour tout $\varphi \in E$ un $s_\varphi < 1$ de manière que $H(s_\varphi, \varphi) > K_1$ où $H(s, \varphi) = \int_{\Gamma(\varphi)} |F(sz, \varphi)|^p |dz|$. Désignons par E_n l'ensemble des points $\varphi \in E$ pour lesquels $s_\varphi \leqslant 1 - 1/n$. Il est évident que $E_n \to E$, ce qui permet de choisir un $n = n_0$ de façon que $|E_{n_0}| > \frac{1}{2}|E|$. En supposant $\varphi \in E_{n_0}$ et en tenant compte du lemme 9, on en conclut que $H(s_\varphi, \varphi) \leqslant AH(1 - 1/n_0, \varphi)$, où A ne dépend ni de φ ni de n_0. Il en vient

$$\tfrac{1}{2}K_1 |E| \leqslant \int_{E_{n_0}} H(s_\varphi, \varphi) \, d\varphi \leqslant AM.$$

Or, K_1 étant arbitraire, on a en conséquence $|E| = 0$, ce qui prouve qu'on a presque partout $H(\varphi) < \infty$.

Ceci établi, choisissons un r_φ de manière que l'on ait $H(r_\varphi, \varphi) > \frac{1}{2}H(\varphi)$ et désignons par E_n l'ensemble des points pour lesquels $r_\varphi < 1 - 1/n$. En tenant compte du lemme 9, on obtient pour $\varphi \in E_n$ l'inégalité $H(r_\varphi, \varphi) \leqslant AH(1 - 1/n, \varphi)$ où $A = A(K, \lambda, p)$; c. à. d.

$$\int_{E_n} H(\varphi) \, d\varphi \leqslant 2AM, \qquad \int_0^{2\pi} H(\varphi) \, d\varphi \leqslant 2AM. \tag{8.2}$$

D'autre part, on conclut du lemme 17 que presque partout

$$\int_{\Gamma(\varphi)} |F(sz, \varphi) - F(z, \varphi)|^p \, |dz| \to 0,$$

ce qui entraîne le lemme en vertu de (8.2).

LEMME 19. *$F(z_1, z_2)$ étant une fonction de classe H_p $(p > 1)$ dans un domaine*

D de type **R**, *il existe une fonction* $F(z_1, e^{i\varphi_2})$ *analytique pour presque tout* φ_2 *et satisfaisant à la condition*

$$\lim_{s\to 1}\int_0^{2\pi}\int_0^{2\pi} |F(rh(se^{i\varphi_2},\lambda), se^{i\varphi_2}) - F(rh(e^{i\varphi_2},\lambda), e^{i\varphi_2})|^p\, d\lambda\, d\varphi_2 = 0 \tag{8.3}$$

$$(0 < r < 1).$$

D'après la définition de la classe H_p, on a pour tout $r < 1$ et $s < 1$ suffisamment proches de 1

$$\int_0^{2\pi}\int_0^{2\pi} |F(rh(se^{i\varphi_2},\lambda), se^{i\varphi_2})|^p\, d\lambda\, d\varphi_2 \leqslant M.$$

En tenant compte du lemme 18, il existe donc une fonction $F(rh(e^{i\varphi_2},\lambda), e^{i\varphi_2})$ telle que:

$$\int_0^{2\pi}\int_0^{2\pi} |F(rh(e^{i\varphi_2},\lambda), e^{i\varphi_2})|^p\, d\lambda\, d\varphi_2 \leqslant AM,$$

$$\lim_{s\to 1}\int_0^{2\pi}\int_0^{2\pi} |F(rh(se^{i\varphi_2},\lambda), se^{i\varphi_2}) - F(rh(e^{i\varphi_2},\lambda), e^{i\varphi_2})|^p\, d\lambda\, d\varphi_2 = 0.$$

Supposons le lemme 19 vrai si le domaine D est un bicylindre quelconque. En tenant compte du lemme 9, la fonction $F(z_1, e^{i\varphi_2})$ est analytique dans presque tout cercle $|z_1| \leqslant \Delta$, $z_2 = e^{i\varphi_2}$ et, à plus forte raison, dans presque tout domaine $D_s(\varphi_2)$ borné par la courbe $\Gamma_s(\varphi_2)$ où $z_1 = sh(z_2, \lambda)$ et $z_2 = e^{i\varphi_2}$. Soit s_0 la borne supérieure de nombres s pour lesquels la fonction $F(z_1, e^{i\varphi_2})$ est encore analytique dans les domaines $D_s(\varphi_2)$ sauf peut-être pour un ensemble de mesure nulle de valeurs de φ_2. Soit E_n l'ensemble de points φ_2 tels que la fonction $F(z_1, e^{i\varphi_2})$ n'est pas analytique dans le domaine $D_{s_0-1/n}(\varphi_2)$. Il est évident que, sauf pour l'ensemble $E = \sum_n E_n$, la fonction $F(z_1, e^{i\varphi_2})$ est analytique dans les domaines $D_{s_0}(\varphi_2)$. Fixons $\lambda = \lambda_0$ et effectuons la transformation

$$z_1' = \frac{z_1 - s_0 h(z_2, \lambda_0)}{ih_\lambda'(z_2, \lambda_0),}, \qquad z_2' = z_2. \tag{8.4}$$

Le transformé D' du domaine D contient un bicylindre de la forme $|z_1'| \leqslant \rho$, $|z_2| \leqslant 1$, à moins que $s_0 \neq 1$. Le lemme 9 montre que la fonction $\Phi(z_1', z_2') = F(z_1, z_2)$, où z_1, z_2 et z_1', z_2' sont liés par les relations (8.4), est de classe H_p. En appliquant donc le lemme 19, supposé vrai pour le bicylindre, on voit que la fonction $\Phi(z_1', e^{i\varphi_2'})$ est analytique dans presque tout domaine $|z_1'| = \rho$, $z_2' = e^{i\varphi_2'}$. En effectuant la représentation inverse à (8.4), on voit que la fonction $F(z_1, e^{i\varphi_2})$ est analytique dans presque tout cercle $|z_1 - s_0 h(z_2, \lambda_0)| = r$, $z_2 = e^{i\varphi_2}$. La même opération peut être faite pour tout les λ, et on voit facilement que la borne inférieure des rayons r correspondants est positive. Il en résulte l'existence d'une suite $\lambda_0, \lambda_1, \cdots, \lambda_n$ telle qu'en désignant par $K(r, \varphi_2, \lambda)$ le cercle $|z_1 - s_0 h(z_2, \lambda)| \leqslant r$, $z_2 = e^{i\varphi_2}$, on a $D_{s_0}(\varphi_2) + \sum_i K(r, \varphi_2, \lambda_i) \supset D_{s_1}(\varphi_2)$ où $s_1 > s_0$ est indépendant de φ_2. On en conclut que la fonction $F(z_1, e^{i\varphi_2})$ est encore analytique dans presque tout domaine $D_{s_1}(\varphi_2)$.

Donc $s_0 = 1$ et la démonstration du lemme 19 s'achève par le lemme suivant

LEMME 20. *$F(z_1, z_2)$ étant une fonction analytique de classe H_p $(p > 1)$ dans le bicylindre $|z_1| \leqslant 1, |z_2| \leqslant 1$, il existe une fonction $F(z_1, e^{i\varphi_2})$ analytique pour presque tout φ_2 et satisfaisant à la condition*

$$\lim_{s\to 1} \int_0^{2\pi}\int_0^{2\pi} |F(re^{i\varphi_1}, se^{i\varphi_2}) - F(re^{i\varphi_1}, e^{i\varphi_2})|^p \, d\varphi_1 \, d\varphi_2 = 0 \qquad (0 \leqslant r < 1).$$

Soit $U(z_1, z_2)$ le partie réelle de $F(z_1, z_2)$. Il existe[16] une fonction $f(\varphi_1, \varphi_2)$ de classe L^p telle que

$$U(z_1, z_2) = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(\varphi_1 + \varphi', \varphi_2 + \varphi'') \cdot \frac{(1 - r_1^2)(1 - r_2^2)\, d\varphi' \, d\varphi''}{(1 - 2r_1 \cos\varphi' + r_1^2)(1 - 2r_2 \cos\varphi'' + r_2^2)}$$

où $z_1 = r_1 e^{i\varphi_1}$ et $z_2 = r_2 e^{i\varphi_2}$. En posant

$$U(z_1, e^{i\varphi_2}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi_1 + \varphi', \varphi_2)\frac{(1 - r_1^2)\, d\varphi'}{1 - 2r_1 \cos\varphi' + r_1^2},$$

on obtient

$$\lim_{s\to 1} \int_0^{2\pi}\int_0^{2\pi} |U(re^{i\varphi_1}, se^{i\varphi_2}) - U(re^{i\varphi_1}, e^{i\varphi_2})|^p \, d\varphi_1 \, d\varphi_2 = 0 \quad (r < 1).$$

La même proposition étant vraie aussi pour la partie imaginaire de $F(z_1, z_2)$, le lemme 20, et partant le lemme 19, se trouve établi.

§9. Le th. 1 à démontrer résulte des lemmes 16–20 comme il suit.

En appliquant les lemmes 18 et 19, il existe une fonction $F(e^{i\varphi_1}, e^{i\varphi_2})$ qui peut être considérée pour presque tout φ_2 comme la fonction frontière de la fonction analytique $F(z_1, e^{i\varphi_2})$. Elle est évidement de classe L^p. En partant d'elle et en effectuant les opérations décrites au §5, nous retrouvons donc la fonction $F(z_1, z_2)$. Par conséquent nous sommes dans les conditions du th. 2 et, celui-ci étant établi, il en est de-même du th. 1.

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TRAVAUX CITÉS

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[2] ——*Zur Veranschaulichung der Kreiskörper und Bereiche mit ausgezeichneter Randfläche.* Jahresbericht der Deutschen Math. Verein. **44** (1933), p. 238–252.

[3] ——*Über eine in gewissen Bereichen mit Maximumfläche gültige Integraldarstellung der Funktionen zweier komplexer Variabler.* II. Math. Zeit. **39** (1934), p. 605–608.

[16] Ce qu'on peut déduire immédiatement des théorèmes connus (voir p. ex. A. Zygmund [12], p. 86–87).

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A REMARK ON THE PAPER "SUR LES FONCTIONS ANALYTIQUES DE DEUX VARIABLES COMPLEXES"[1]

By Stefan Bergman

1. One of the problems of the theory of summation consists in finding for a given system of O.N. (orthonormal) functions $\{\Psi_\nu(\varphi, \lambda)\}$, $0 \leqq \varphi \leqq 2\pi$, $0 \leqq \lambda \leqq 2\pi$, a system of functions ψ_ν depending on φ, λ and a parameter r, such that $\lim_{r\to 1} \psi_\nu = \Psi_\nu$ and possessing the property that for every function $f \in L^2$ $f(\varphi, \lambda) \in L^2$, $\lim_{r\to 1} \Sigma a_\nu \psi_\nu = f$ almost everywhere, $a_\nu = \int_0^{2\pi}\int_0^{2\pi} f\Psi_\nu \, d\varphi \, d\lambda$. The results of the paper cited in the title and some other related papers yield such a method which in the case where $\{\Psi_\nu\}$ forms the system $\{\cos m\varphi \cos n\lambda, \cos m\varphi \sin n\lambda, \sin m\varphi \cos n\lambda, \sin m\varphi \sin n\lambda\}$ becomes the Poisson's summation method.

2. We consider at first the case of one variable. Let $\{\Psi_\nu(\varphi)\}, 0 \leqq \varphi \leqq 2\pi$, be a set of continuous O.N. functions which are complete in L^2. That is to say, we have

$$(2.1) \qquad \int_0^{2\pi} \Psi_\nu(\varphi)\Psi_\mu(\varphi)\, d\varphi = \delta_{\nu\mu}, \qquad \delta_{\nu\mu} = 0 \text{ for } \nu \neq \mu, \ \delta_{\nu\nu} = 1,$$

and for every $H(\varphi) \in L^2$

$$(2.2) \qquad \lim_{n\to\infty} \int_0^{2\pi} \left[H(\varphi) - \sum_{\nu=1}^{n} a_\nu \Psi_\nu(\varphi) \right]^2 d\varphi = 0, \qquad a_\nu = \int_0^{2\pi} H(\varphi)\Psi_\nu(\varphi)\, d\varphi.$$

By $\psi_\nu(r, \varphi)$ we denote the system of harmonic functions possessing the boundary values $\Psi_\nu(\varphi)$, that is we have $\lim_{r\to 1} \psi_\nu(r, \varphi) = \Psi_\nu(\varphi)$.

[1] Published in this Journal, this vol., p. 125. For notations used hereafter we refer to this paper.

Then: *1.° the series* $\sum_{\nu=1}^{\infty} a_\nu \Psi_\nu$ *converges uniformly and absolutely for* $r < 1$; *2.°* $\lim_{r\to 1} \sum_{\nu=1}^{\infty} a_\nu \psi_\nu(r, \varphi) = H(\varphi)$ *almost everywhere.*

Proof. 1. At first we shall show that $\sum_{\nu=1}^{\infty} [\psi_\nu(r, \varphi)]^2 < \infty$. If we determine the minimum λ of $\int_0^{2\pi} \left(\sum_{\nu=1}^{n} X_\nu \Psi_\nu(\varphi)\right)^2 d\varphi$ under the condition that

$$\sum_{\nu=1}^{n} X_\nu \Psi_\nu(r^0, \varphi^0) = 1$$

a formal calculation yields $\lambda = \left[\sum_{\nu=1}^{n} (\psi_\nu(r^0, \varphi^0))^2\right]^{-1}$. On the other hand for every harmonic function $G(r, \varphi)$ continuous for $r = 1$, we have

$$2\pi \,|\, G(r^0, \varphi^0) \,| \leqq \frac{1 + r^0}{1 - r^0} \sqrt{2\pi \int_0^{2\pi} G^2(1, \varphi)\, d\varphi}.$$

Thus:

$$1 \leqq \frac{1}{2\pi}\left(\frac{1 + r^0}{1 - r^0}\right)^2 \lambda = \frac{1}{2\pi}\left(\frac{1 + r^0}{1 - r^0}\right)^2 \left[\sum_{\nu=1}^{n} (\psi_\nu(r^0, \varphi^0))^2\right]^{-1}.$$

2. Let $P(r, \varphi)$ be a harmonic function posessing the boundary values $H(\varphi)$. From:

$$2\pi \left| P(r, \varphi) - \sum_{\nu=1}^{n} a_\nu \psi_\nu(r, \varphi) \right| \leqq \frac{1 + r}{1 - r} \int_0^{2\pi} \left| H(\varphi) - \sum_{\nu=1}^{n} a_\nu \Psi_\nu(\varphi) \right| d\varphi$$

$$\leqq \frac{1 + r}{1 - r} \sqrt{2\pi \int_0^{2\pi} \left[H(\varphi) - \sum_{\nu=1}^{n} a_\nu \Psi_\nu(\varphi)\right]^2 d\varphi}$$

and (2.2), follows $P(r, \varphi) = \sum_{\nu=1}^{\infty} a_\nu \psi_\nu(r, \varphi)$, for $r < 1$. Since by Fatou's theorem $\lim_{r\to 1} P(r, \varphi) = H(\varphi)$, 2^0 follows.

3. The above procedure can be generalized to the case of two variables. If the domain of definition of $\{\varphi, \lambda\}$ is the sphere we use harmonic functions of three variables. If the domain of definition is homeomorphic to a torus we may use either harmonic functions of three variables or the functions of the "extended class"[2] defined in a domain D of type **R** with a distinguished boundary surface. (For details see the papers mentioned in the title and in the footnote [2].)

We may repeat in the case of functions of the extended class the same considerations as in the section 2. The results of Jessen, Marcinkiewicz, Zygmund[3]

[2] See Bergman, "Über meromorphe Funktionen von zwei komplexen Veränderlichen," Compositio Math. vol. **6** (1939) p. 305–335, especially p. 319.

[3] "Note on the differentiability of multiple integrals," Fund. Math. vol. **25** (1935), p. 217–234.

(if D is a bicylinder) and the result of the above cited paper and its generalization obtained by Bers[4] assure (in the general case) the existance almost everywhere of boundary values on the distinguished boundary surface for every function $P(r_1, r_2, \varphi, \lambda) \in H_n$, $n > 1$, of the extended class. Thus if we determine for a system of O.N. functions $\Psi_\nu(\varphi, \lambda)$, $0 \leqq \varphi \leqq 2\pi$, $0 \leqq \lambda \leqq 2\pi$, the functions of the extended class $\psi_\nu(r_1, r_2, \varphi, \lambda)$ possessing on the distinguished boundary surface the boundary values $\Psi_\nu(\varphi, \lambda)$ we have, analogously as above, for every $H \in L^2$, $\lim \sum_{\nu=1}^{\infty} a_\nu \psi_\nu(r_1, r_2, \varphi, \lambda) = H(\varphi, \lambda)$, where $a_\nu = \int_0^{2\pi} \int_0^{2\pi} H(\varphi, \lambda)\Psi_\nu(\varphi, \lambda)\, d\varphi\, d\lambda$, almost everywhere.

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[4] "Sur les valeurs limites des fonctions analytiques de deux variables complexes dans les domaines possédant une surface frontière remarquable." C. R. Acad. Sci., Paris vol. 208 (1939) p. 1273–1275, 1475–1477, and "Bounded analytic functions of two complex variables in certain domains with distinguished boundary surface" will appear in Am. Journ. of Math.

RULED SURFACES IN EUCLIDEAN FOUR SPACE[1]

By Martha Hathaway Williams

I. This paper summarizes a detailed investigation of the vectors and tensors associated with a ruled surface imbedded in an Euclidean space of four dimensions. The fundamental forms of the surface and important sets of curves on the surface are presented with particular emphasis on the unique curves, the striction curve and the quasi-asymptotic curve of the first kind. Curvature properties are studied by considering the variation of the normal vectors and the curvature conic along the generator of the surface. Isotropic surfaces and the subject of the applicability of ruled surfaces are also presented.

The notation used is a composite of ordinary vector and tensor notations, where, for example,

$$a_{\kappa\lambda}v^{\kappa}w^{\lambda} = v^{\kappa}w_{\kappa} = v_{\lambda}w^{\lambda}$$

is the scalar product. In the case of Euclidean spaces this may be written

$$v^{\kappa}w^{\kappa} \text{ or } \bar{v}\cdot\bar{w}$$

The formal manipulation of tensors follows the algebra developed in Schouten and Struik I.[2] Numbers in footnotes refer to the bibliography at the end of the paper, which is arranged in chronological order.

II. A ruled surface, S, in an Euclidean space of four dimensions, R_4, may be considered as generated by a vector moving along a twisted space curve.[3] If the curve, C, is represented by $x^{\kappa}(t)$ and the moving vector by $i^{\kappa}(t)$, $(\kappa = 1, 2, 3, 4)$, where the functions of the parameter t are sufficiently regular to permit differentiation as may be required, any point, P, on the surface with coordinates y^{κ}, $(\kappa = 1, \cdots, 4)$ is given by

$$y^{\kappa}(t, u) = x^{\kappa}(t) + ui^{\kappa}(t), \qquad \kappa = 1, 2, 3, 4 \tag{1}$$

or

$$\bar{y}(t, u) = \bar{x}(t) + u\bar{i}(t)$$

If $\bar{i}(t)$ is a unit vector, $i^{\kappa}i_{\kappa} = 1$, u is the distance of P from C measured along the ruling. The curve C is the directrix of the surface. $\bar{i}$ is a generator. If the vectors $\bar{i}$ are moved to one point, the $i^{\kappa}(t)$ cut the unit hypersphere on the same origin in a curve, and form the director cone.

Since the surrounding space is Euclidean, distance is measured as

$$ds^2 = dy^{\kappa}dy_{\kappa}$$

[1] Most of this paper was presented to the Mathematics Department of M.I.T. as a thesis in partial fulfillment of the requirements for the Ph.D. degree. The subject was suggested by Professor D. J. Struik, who directed the work, and to him I wish to express my sincere thanks and appreciation. The manuscript was completed in March, 1940.

[2] Schouten and Struik. 1935: 3.

[3] W. C. Graustein: 1935: 1. p. 208.

which induces on S a metric

$$ds^2 = g_{ij}dx^i dx^j, \qquad (i, j, = 1, 2) \tag{2}$$

where $x^1 = t, \quad x^2 = u, \quad g_{11} = \bar{x}_t \cdot \bar{x}_t + 2u\bar{\imath} \cdot \bar{x}_t + u^2 \bar{\imath}_t \cdot \bar{\imath}_t$

$$g_{12} = g_{21} = \bar{\imath} \cdot \bar{x}_t$$

$$g_{22} = \bar{\imath} \cdot \bar{\imath}$$

If $\bar{\imath}$ is a unit vector and t is the arc length of C, (2) becomes

$$ds^2 = (1 + 2u\bar{\imath}_t \cdot \bar{x}_t + u^2 \bar{\imath}_t \cdot \bar{\imath}_t)\, dt^2 + 2\bar{\imath} \cdot \bar{x}_t dt\, du + du^2 \tag{3}$$

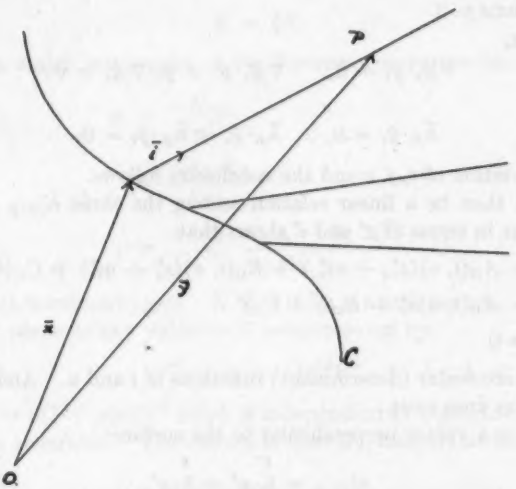

FIGURE I

where $\bar{\imath} \cdot \bar{x}_t = \cos \vartheta$, where ϑ is the angle between C and the ruling. (It will sometimes be convenient also to use the notation $N = \bar{\imath}_t \cdot \bar{x}_t$ and $M^2 = \bar{\imath}_t \cdot \bar{\imath}_t$.) The orthogonal trajectories of the rulings are the integral curves of[4]

$$\cos \vartheta\, dt + du = 0$$

which can be solved by quadratures. *A given ruled surface in an Euclidean space of four dimensions can always be transformed so the directrix is an orthogonal trajectory of the rulings and distance on the surface is measured by*

$$ds^2 = (1 + 2\bar{\imath}_t \cdot \bar{x}_t u + u^2 \bar{\imath}_t \cdot \bar{\imath}_t)\, dt^2 + du^2 \tag{4}$$

[4] L. P. Eisenhart, 1909: 1.

Designating by h^κ_{ij} the second covariant derivatives of the functions defining the surface,[5] we have

$$h^\kappa_{ij} = y^\kappa_{ij} - \Gamma^c_{ij} y^\kappa_c, \qquad \kappa = 1, 2, 3, 4, \qquad i, j, c = 1, 2 \tag{5}$$

where
the Γ^c_{ij} are Christoffel symbols of the second kind. Since $h^\kappa_{ji} = y^\kappa_{ji} - \Gamma^c_{ji} y^\kappa_c$ and $y^\kappa_{ij} = y^\kappa_{ji}$ and the Γ^c_{ij} are symmetric, h^κ_{ij} *is a symmetric tensor with respect to the covariant indices.*

With respect to the contravariant index κ, *the* $h^\kappa_{(i)(j)}$ *are vectors in the* R_4 *normal to the surface and lie in the plane which is completely perpendicular to the tangent plane at any point.*[6]

$$\bar{y}_i \cdot \bar{y}_j = 0, \qquad \bar{y}_i \cdot \bar{y}_i = 1$$

from assumptions p. 2.
Differentiating,

$$\nabla \bar{y}_i \cdot \bar{y}_i = 0, \qquad \nabla \bar{y}_i \cdot \bar{y}_c + \bar{y}_i \cdot \nabla \bar{y}_c = 0$$

and

$$\bar{h}_{ji} \cdot \bar{y}_i = 0, \qquad \bar{h}_{ji} \cdot \bar{y}_c + \bar{h}_{jc} \cdot \bar{y}_i = 0, \qquad i, j, c = 1, 2$$

for any permutation of i, j, c and the conclusion follows.

There must then be a linear relation among the three $h^\kappa_{(i)(j)}$. Calculation of these vectors in terms of x^κ and i^κ shows that

$$\begin{aligned} h_{11} &= A_{11}(t, u)(x^\kappa_{tt} + u i^\kappa_{tt}) + B_{11}(t, u)(x^\kappa_t + u i^\kappa_t) + C_{11}(t, u) i^\kappa \\ h_{12} &= A_{12}(t, u) x^\kappa_t + B_{12} i^\kappa_t + C_{12} i^\kappa \\ h_{22} &= 0 \end{aligned} \tag{6}$$

where the A_{ij} are scalar (determinant) functions of t and u. And perforce such a linear relation does exist.

Since $h^\kappa_{(i)(j)}$ is a vector perpendicular to the surface

$$h^\kappa_{(i)(j)} = \overset{1}{h}_{ij} p^\kappa + \overset{2}{h}_{ij} q^\kappa, \qquad \kappa = 1, 2, 3, 4 \tag{7}$$

where $\overset{\alpha}{h}_{ij}$ are scalar functions of t and u and p^κ and q^κ are unit orthogonal vectors in the normal plane to the surface.[7] Then

$$\bar{h}_{ij} \cdot \bar{h}_{ij} = \sum_\alpha \overset{\alpha}{h}_{ij} \overset{\alpha}{h}_{ij}, \qquad \alpha = 1, 2$$

and the Gauss-Codazzi-Ricci equations for the surface, S, are

$$\begin{aligned} R_{1212} &= \bar{h}_{12} \cdot \bar{h}_{12} \\ \nabla \overset{1}{h}_{[i\ j]c} &= \overset{1\ 2}{v}h_{2[i\ j]c}, \qquad \nabla \overset{2}{h}_{[i\ j]c} = \overset{2\ 1}{v}h_{1[i\ j]c} \\ \nabla \overset{1}{v}_{[i2j]} &= g^{ab}(\overset{1}{h}_{a[i1b1j]} \overset{2}{h}) \end{aligned} \tag{8}$$

[5] L. P. Eisenhart, 1926: 1, p. 160.
[6] The $(i)(j)$ indicate the indices are "dead."
[7] This can always be done. Cf. Eisenhart 1926: 1, p. 143.

where R_{1212} is the only non-zero component of the Riemann curvature tensor of S, $v_{21}^{1} = -v_{11}^{2} = p^{\kappa}\nabla_i q^{\kappa}$, and ∇ indicates covariant differentiation with respect to S.

The second fundamental form of the surface is a vector quadratic differential form defined in terms of h_{ij}^{κ}.

$$\psi^{\kappa} = h_{ij}^{\kappa}\, dx^i\, dx^j, \qquad i, j = 1, 2 \tag{9}$$

or

$$\psi^{\kappa} = h_{11}^{\kappa}\, dt^2 + 2h_{12}^{\kappa}\, dt\, du. \tag{10}$$

The Gaussian curvature of a ruled surface in R_4 is never positive. The Gaussian curvature is defined as

$$K = \tfrac{1}{2}R$$

where R is the scalar component of the Riemann curvature tensor. Whence,

$$\begin{aligned} K &= \frac{-R_{1212}}{|g|} \\ &= \frac{-\bar{h}_{12}\cdot\bar{h}_{12}}{|g|} \\ &= \frac{-(\sin^2\vartheta\bar{v}_1\cdot\bar{v}_1 - (\bar{v}_1\cdot\bar{x}_1)^2)}{|g^2|} \end{aligned} \tag{11}$$

where g is the determinant $|g_{ij}|$. $K = 0$ if $\bar{h}_{12}$ is a null vector.

The tangent plane at any point on S is represented by

$$T^{\kappa\lambda} = (x_i^{[\mu} + ui_i^{[\mu})i^{\nu]}, \qquad \kappa, \lambda, \mu, \nu = 1, 2, 3, 4. \tag{13}$$

If $K = 0$, $T_0^{\kappa\lambda} = a(1 + u)x_i^{[\mu}i^{\nu]}$ which is independent of u and the tangent plane is fixed along a generator. The surface is then the tangent surface of a twisted curve.

If $K = 0$ defines developable surfaces, then *ruled surfaces which are developable are twisted curve surfaces, and conversely.* But there exist in R_4 developable surfaces which are not ruled,[8] since

$$K = \frac{\bar{h}_{11}\cdot\bar{h}_{22} - \bar{h}_{12}\cdot\bar{h}_{12}}{|g|}$$

for any surface, may vanish without $h_{22}^{\kappa} = h_{12}^{\kappa} \equiv 0$.

There are no ruled surfaces of constant Gaussian curvature. This follows from (12).

For any regular generator K will have a minimum value at the point

$$u = u_0 = \frac{-\bar{v}_1\cdot\bar{x}_1}{\bar{v}_1\cdot\bar{v}_1} \tag{14}$$

found by differentiating (12) with respect to u and equating to zero.

[8] Wilson, 1916: 1, p. 343.

$u = u_0$ also defines the striction curve of the surface, where the striction curve is the locus of the striction points of the generators. The striction point of a ruling is defined as the point of minimum distance from a consecutive ruling. That is, if the minimum distance between rulings t and $t + dt$ is sought, two equations are found, namely

$$\begin{aligned} &\frac{\partial(ds^2)}{\partial(du)} = 2\cos\vartheta\, dt + 2\, du = 0 \\ &\frac{\partial(ds^2)}{\partial(u)} = (2\bar{\imath}_i \cdot \bar{x}_i + 2\bar{\imath}_i \cdot \bar{\imath}_i u)\, dt^2 = 0 \end{aligned} \tag{15}$$

which admit $u = u_0$ as a solution.[9]

The striction curve is, then, the locus of minimum Gaussian curvature.

The tangent planes along a ruling all lie in the same R_3 which from (13) has covariant components

$$v_\lambda = e_{\lambda\kappa\mu\nu} x^\kappa_i \bar{\imath}^\mu_i \bar{\imath}^\nu \tag{16}$$

where $e_{\lambda\kappa\mu\nu} = \pm 1$ depending on the permutation of the indices.[10] The normal vector can be written

$$v^\lambda = e^\lambda_{\cdot\kappa\mu\nu} x^\kappa_i \bar{\imath}^\mu_i \bar{\imath}^\nu, \qquad (\lambda, \kappa, \mu, \nu = 1, 2, 3, 4.) \tag{17}$$

Then the angle φ between the tangent plane at a point u and the tangent plane at u_0 can be shown to be a linear function of the distance between the two points

$$\tan\varphi = \frac{|u - u_0|}{|v|}\,[11] \tag{18}$$

The tangent planes rotate around the rulings.

The above properties of the striction curve will be referred to later in the paper. Other properties of the curve for S in R_3 will be found discussed in the literature and can be shown to carry over to S in R_4.[12]

III. Any non-minimal curve on the surface has an absolute curvature vector[13]

$$\begin{aligned} n^\kappa &= \frac{dy^\lambda}{ds}\, {}'\nabla_\lambda \frac{dy^\kappa}{ds} \\ &= \frac{\partial y^\kappa}{\partial x^j}\frac{dx^i}{ds}\nabla_i \frac{dx^j}{ds} + h^\kappa_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} \end{aligned} \tag{19}$$

where

$${}'n^\kappa = \frac{\partial y^\kappa}{\partial x^j}\frac{dx^i}{ds}\nabla_i \frac{dx^j}{ds} \tag{20}$$

[9] Kommerell, 1905: 1.
Kühne, 1901: 1.
[10] McConnell, 1936: 2.
[11] Cf. Graustein, 1935: 1, p. 211.
[12] O. Bonnet, 1867: 1; E. Bour, 1862: 1, 2.; H. Beck, 1928: 1; J. Krames, 1928: 2; A. Myller, 1922: 1.
[13] Schouten, 1938: 1, p. 80.

is the relative curvature vector of the curve, or the component of the curvature in the surface, and

$$''n^\kappa = h^\kappa_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} \tag{21}$$

is the normal curvature vector or the component of the curvature perpendicular to the surface.

All curves through a point P on the surface in the same direction have the same normal curvature. This follows from the fact that the magnitude of n^κ can be written

$$\frac{1}{r_1^2} = \frac{1}{\rho^2} + \frac{1}{R^2} \tag{22}$$

where R is the radius of curvature of the geodesic, ρ that of the curve with respect to S and r_1 that of the curve with respect to R_4 in a given direction.

By comparing (21) and (9), the second form of the surface is the normal curvature vector for a direction at a point on the surface.

The mean curvature vector at a point P is

$$M^\kappa = \frac{1}{2g} (h^\kappa_{11} - \cos \vartheta h^\kappa_{12}) \tag{23}$$

for the surface (3) or, for (4)

$$M^\kappa = \frac{1}{2g} h^\kappa_{11} \tag{24}$$

S is a minimal surface, in the sense of least area, if M^κ is a null vector.[14] But this requires

$$h^\kappa_{11} = \cos \vartheta h^\kappa_{12}$$

which in general is not true.

Then, *the only minimal ruled surfaces in R_4 are those of R_3.*

Since $''n^\kappa$ is perpendicular to the surface at any point for all directions at that point, it lies in the normal plane to the surface at that point. As the direction changes, the end point of $''n^\kappa$ traces a curve in this plane which is a finite, closed curve so long as we treat only real directions. If S is referred to orthogonal parameters then with respect to these parameters any direction $dt:du$ will be represented by

$$i^a = \cos \alpha \underset{1}{i^a} + \sin \alpha \underset{2}{i^a} \tag{25}$$

where $\underset{1}{i^a}:0, 1; \underset{2}{i^a}:1/\sqrt{g}, 0 \quad (a = 1, 2.)$

and $''n^\kappa$ for i^a is given by

$$''n^\kappa = i^a i^b h^\kappa_{ab} = M^\kappa + \sin 2\alpha \underset{1}{\overset{2}{''n^\kappa}} - \tfrac{1}{2} \cos 2\alpha \underset{2}{''n^\kappa} \tag{26}$$

[14] Wilson, 1916: 1, p. 325.

where

$$''n^\kappa_1 = 0, \qquad ''n^\kappa_2 = \frac{1}{g} h^\kappa_{11}, \qquad ''n^\kappa_1 = (1/\sqrt{g}) h^\kappa_{12}.$$

The normal curvature in the direction of the rulings is zero. The curve (26) *is a conic with center* M^κ *and* $\frac{1}{2}''n^\kappa_2$, $''n^\kappa_1$ *as a pair of conjugate radii and it passes through the surface point* P.[15]

The normal planes to a ruled surface along a generator rotate about a fixed direction in the space R_4 *in such a way that the tangent of the angle between the normal plane at a point* $P(t, u)$ *and the normal plane at the striction point* $P_0(t, u)$ *is a linear function of the distance* $|u - u_0|$, *and the angle is equal to that between the tangent planes at the two points.*

The vectors p^κ and q^κ ($\kappa = 1, 2, 3, 4$) of (7) which define the normal planes are determined only to within a rotation,[16] so if p^κ is taken as the vector v^κ (17) normal to the R_3 tangent to S along a generator it depends only on t and not on u and $q^\kappa(t, u)$ then, as it varies with u, will exhibit the change in the normal planes. At the directrix, C, $[x^\alpha_i i^\nu_i]$ is a plane perpendicular to i^κ and with it in the tangent R_3. q^κ and y^κ_i are parallel to this plane for all u. $[y^\alpha_i i^\nu]$ is the tangent plane, T, which rotates with angle φ given by (18). $[q^\alpha i^\nu]$ defines a family of planes $[Q]$ in one-to-one correspondence with the family $[T]$ and corresponding planes are perpendicular. Then the angle between the planes of $[Q]$ at P and P_0 is φ; or the angle between $q^\kappa(t, u)$ and $q^\kappa(t, u_0)$ is φ. $[p^\alpha q^\nu]$ defines the family of normal planes, $[N]$. As u varies N rotates around p^κ, and since q^κ is parallel to a plane perpendicular to p^κ, the angle between N and N_0 is that between q^κ and q^κ_0, namely, φ.

If the line in the direction of i^κ is the x-axis, that in the direction p^κ the y-axis, a plane perpendicular to i^κ is represented by

$$x = \lambda$$
$$w = ay + bz.$$

This plane, to be N, must contain the line $x = \lambda$, $w = z = 0$. Thus N is given by

$$\begin{cases} x = \lambda \\ w = bz. \end{cases}$$

Since N rotates with angle φ there is a linear relation between λ and b, say $b = k\lambda$, where k is a constant. (The axis frame has been set up at P_0.) Thus the locus of N is

$$w = kxz$$

which is a cylinder formed by lines parallel to the y-axis in points of an hyperbolic parabaloid in the wxz-space. (See figures 2 and 3.)

[15] The curvature properties may also be studied by means of an extended Dupin indicatrix, the curve in which an R_3 perpendicular to M^κ cuts S.

[16] Wilson, 1916: 1, p. 309.

Conjugate directions and asymptotic directions at a point of S coincide with and are in the direction of the ruling through the point. By definition they are the directions which satisfy the equation

$$h^\epsilon_{11}\,dt\,\delta t + h^\epsilon_{12}(dt\,\delta u + \delta t\,du) = 0 \tag{27}$$

or

$$h^\epsilon_{11}\,dt^2 + 2h^\epsilon_{12}\,dt\,du = 0. \tag{28}$$

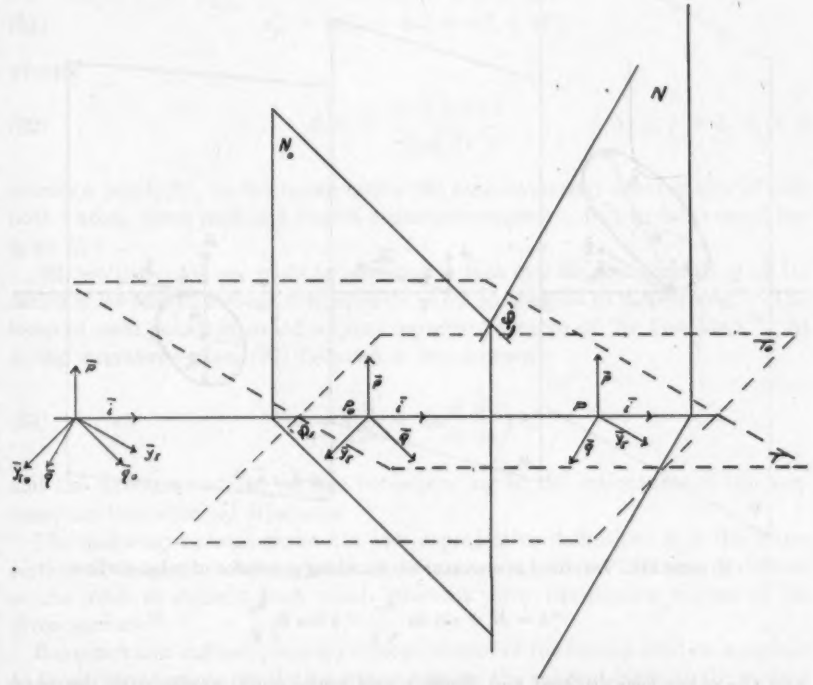

FIGURE II. Normal planes along a generator of a ruled surface in R_4

At any point for which $h^\epsilon_{11} \neq kh^\epsilon_{12}$ then $dt = 0$, $\delta t = 0$ are the only solutions. At such a point there is only one real asymptotic direction.

The directions on the surface in which the normal curvature has maxima and minima magnitudes are defined as the principal directions at a point of the surface. They are given as the roots of the equation

$$\lambda\{-4g_{11}h^\epsilon_{11}h^\epsilon_{12}\lambda^3 + (4h^\epsilon_{11}h^\epsilon_{11} - 8g_{11}h^\epsilon_{12}h^\epsilon_{12})\lambda^2 + 12h^\epsilon_{11}h^\epsilon_{12}\lambda + 8h^\epsilon_{12}h^\epsilon_{12}\} = 0 \tag{29}$$

where

$$\lambda = dt{:}du.$$

The lines of curvature on the surface are the integral curves of (29). At least two of the roots of (29) are real since there is at least one change of sign of the coefficients in the polynomial in brackets. Applying a criterion for equal roots of a cubic,[17] we find that the directions are in general distinct. When h^κ_{11} is perpendicular to h^κ_{12} the parametric directions are two of the principal directions

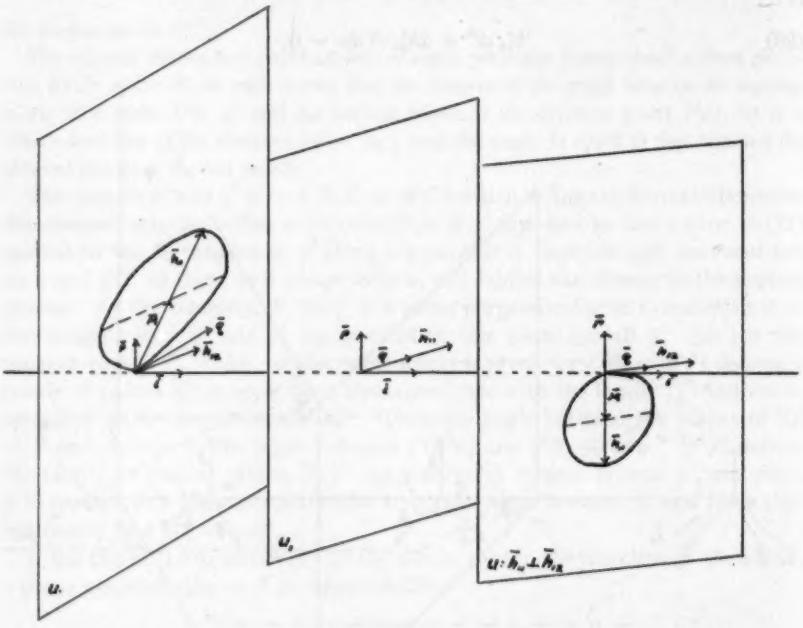

FIGURE III. Variation in curvature conic along generator of ruled surface

$$''\kappa = \bar{M} + \sin 2a \frac{\bar{h}_{12}}{\sqrt{g}} - \tfrac{1}{2} \cos 2a \frac{\bar{h}_{11}}{g}$$

and the other two are real and distinct and make equal angles with the parametric directions if

$$2g_{11}h^\kappa_{12}h^\kappa_{12} > h^\kappa_{11}h^\kappa_{11}\,.$$

At a regular point both parametric curves cannot be lines of curvature since $\lambda = \infty$ does not satisfy (29). To determine whether any two of the other three directions are orthogonal, we must solve

$$\underset{\alpha\beta}{\lambda\lambda} + 1 = 0, \qquad \alpha, \beta = 2, 3, 4, \qquad \underset{1}{\lambda} = dt{:}du = 0.$$

One pair of principal directions may be orthogonal, but *all the principal directions cannot be orthogonal in pairs.*

[17] Cajori, 1904: 1.

If the set of equations

$$h_i^\kappa \rho_\kappa = 0, \qquad (i, j = 1, 2; \kappa = 1, 2, 3, 4) \tag{30}$$

has two linearly independent solutions, h_{ij}^κ is of rank two and a point P of S where this holds is called a planar point.[18] Any point where h_{ij}^κ is of rank one is an axial point.

There is one axial point on each ruling of a ruled surface in R_4. (30) has only one solution if $h_{11}^\kappa = \rho h_{12}^\kappa$. From (6) this obtains if

$$x_{11}^\kappa + u i_{11}^\kappa = a x_1^\kappa + b i_1^\kappa + c i^\kappa \tag{31}$$

whence

$$u = u_1 \frac{-x_{11}^{[\kappa} x_1^\lambda i_1^\mu i^{\nu]}}{i_{11}^{[\kappa} x_1^\lambda i_1^\mu i^{\nu]}}, \qquad \kappa, \lambda, \mu, \nu = 1, 2, 3, 4 \tag{32}$$

locates a point, P_1, on the ruling unless the numerator and denominator of (32) both vanish, since each is a fourth order determinant. But in this case S lies in an R_3.

(32) says: *there is one point on each ruling such that the first normals of all the curves of the surface through that point lie in the* R_3 tangent to the surface.[19] The locus of such points is called a quasi-asymptotic curve of the first kind.[20] At u_1 the curvature conic (26) becomes a line segment

$$''n^\kappa = \left(\frac{d^2}{ds} + 2\rho \frac{dt}{ds}\frac{du}{ds}\right) h_u^\kappa \tag{33}$$

and the directions on the surface corresponding to the extremities of this segment are the principal directions.

The quasi-asymptotic curve has also a projective definition: it is the intersection of the given ruled surface and its transversal, where the latter is defined as the locus of straight lines which intersect three consecutive rulings of the given surface.[21]

Bompiani also defines quasi-asymptotic curves of the second kind on a surface as those curves whose osculating spaces contain the tangent plane to the surface and which have four point contact with the curve.[22] This is expressed in our notation by

$$c_{\kappa\lambda\mu\nu} w^\kappa \frac{dy^\lambda}{ds}\frac{d^2y^\mu}{ds^2}\frac{d^3y^\nu}{ds^3} = 0 \tag{34}$$

where $w^\kappa = \alpha(x_i^\kappa + u i_i^\kappa) + \beta i^\kappa$.

[18] Schouten, 1938: 1.
[19] Cf. (32) and (17).
[20] Bompiani, 1914: 1; Hlavaty, 1924: 1.
[21] Ranum, 1912: 1; 1915: 1, 2; Stouffer, 1927: 1.
[22] Bompiani, 1914: 1; Lane, 1932: 1.

Breaking (34) into sums of determinants of the functions defining S we find a second order differential equation

$$P_1\frac{d^2u}{dt^2}+P_2\left(\frac{du}{dt}\right)^2+P_3\frac{du}{dt}+P_4=0 \tag{35}$$

where P_i are polynomials in u of degree ≤ 3.
This means there are ∞^2 quasi-asymptotic curves of the second kind on a ruled surface in R_4.

IV. The problem of deforming a ruled surface in R_4 into another has been partly solved. The methods used must of necessity be different from those for similar surfaces in R_3 where the arguments are based on a consideration of asymptotic lines—non-existent for surfaces in higher spaces.[23] One way to approach the problem is to find the complete solution of the Gauss-Codazzi-Ricci equations (8). Another is to find a transformation

$$t'(t,u),\ u'(t,u)$$

such that the surface (3) transforms into a ruled surface, S', with the same ds^2 and Gaussian scalar curvature. If U' is a general function of t and u, the rulings of S may correspond to curves of S'; and this problem has not been answered. But if $u'=u'(u)$ the rulings of the S and S' correspond and any two ruled surfaces may be deformed so that rulings remain straight lines, and in ∞^3 ways. The problem is that of solving the five partial differential equations in eight unknowns:[24]

$$\begin{gathered} N(t)=i_i^\kappa x_i^\kappa,\qquad M^2(t)=i_i^\kappa i_i^\kappa,\qquad 1=x_i^\kappa x_i^\kappa \\ 1=i^\kappa i^\kappa,\qquad i^\kappa x_i^\kappa=\cos\vartheta \qquad\qquad \kappa=1,2,3,4. \end{gathered} \tag{37}$$

If $i^\kappa(t)$ is any curve on the unit hypersphere,

$$i^\kappa(t):\ \cos\omega,\ \sin\omega\cos\varphi,\ \sin\omega\sin\varphi\cos\psi,\ \sin\omega\sin\varphi\sin\psi \tag{38}$$

the second condition of (37) determines t.

$$\int_0^\tau M\,dt=\text{arc length of } i^\kappa \text{ on unit sphere} \tag{39}$$

then

$$M^2=\left(\frac{d\omega}{dt}\right)^2+\left(\frac{d\varphi}{dt}\right)^2\sin^2\omega+\sin^2\omega\sin^2\varphi\left(\frac{d\psi}{dt}\right)^2 \tag{40}$$

determines ψ as a function of ω and φ and the last three equations of (37) determine x_i^κ with one arbitrary component. x^κ are determined by quadratures.

$$x_i^\kappa=\cos\vartheta\,i^\kappa+\frac{N}{M^2}i_i^\kappa\pm\frac{\sqrt{M^2\sin^2\vartheta-N^2}}{M}a^\kappa,\qquad \kappa=1,2,3,4 \tag{41}$$

where a^κ is a unit vector orthogonal to i^κ and i_i^κ.

[23] Darboux, 1894: 1; 1896: 1; Bianchi, 1899: 1.
[24] See (3).

This may be interpreted as follows: Any curve C on the unit hypersphere determines the parameter t. The polar surface of C on the hypersphere is also determined. Any curve C' on this polar surface will be a locus of a^x. To any curve C and C' on its polar surface are associated two ruled surfaces. *Any two polar curves on the unit hypersphere determine two ruled surfaces.*

A ruled surface in R_4 can be bent so C and C' are arbitrary polar curves on the unit hypersphere.

All S with the same C but different C' are parallel in R_4.

Following Beltrami[25] we ask what form the directrix may take by bending the surface and then the problem of determining a space curve from its intrinsic equation arises.[26] If ω is the angle between the ruling and the tangent vector to the directrix, ϑ, (3), φ the angle between the tangent plane to the surface and the osculating plane of the surface and ψ the angle between the tangent plane and the plane $[\bar{x}_1\bar{n}_3]$, (38) defines i^x in terms of the principal normals of the directrix. In the process of bending the surface, ω and the relative curvature of the directrix remain invariant since they depend only on the first form of the surface.

$$\frac{d}{dt}(\bar{\imath}\cdot\bar{x}_1) = \bar{\imath}\cdot\frac{d\bar{x}_1}{dt} + \bar{\imath}_1\cdot\bar{x}_1 - \sin\omega\frac{d\omega}{dt} = \sin\omega\frac{\cos\sigma}{r_1} + N. \tag{42}$$

Substituting (42) in (38) gives a functional relation

$$F\left(t, r_1, \frac{dr_1}{dt}, r_2, r_3, \psi, \frac{d\psi}{dt}\right) = 0 \tag{43}$$

which determines ψ as a function of one arbitrary parameter, where r_j is the jth curvature of the directrix. $(j = 1, 2, 3.)$

As special results we have:

1. *A ruled surface can be deformed so a geodesic becomes a straight line and the director cone has a fixed projection on an R_3 in an ∞^1 ways.*
2. *A ruled surface can be deformed so a geodesic becomes a straight line and the director cone cuts the hypersphere in a great circle in just one way.*
3. *Any ruled surface can be deformed so the directrix is a plane curve and the director cone is arbitrary.*
4. *Any ruled surface can be deformed so the directrix becomes the quasi-asymptotic curve of the surface and the director cone is arbitrary.*

V. Lense has shown that the only completely ametric X_n, ($ds^2 = 0$ for all directions) imbedded in an Euclidean R_{2n} are the linear spaces.[27] So the only completely ametric surfaces in R_4 are the completely isotropic planes. A ruled surface then may contain fields of isotropic vectors, but even so a metric will be defined and general surface theory arguments can be carried through.

[25] 1865: 1, 2.
[26] Forsythe, 1930: 1.
[27] 1926: 2.

If the generators are isotropic, $\bar{j}\cdot\bar{j} = 0$, the ds^2 for the surface $y^\kappa = x^\kappa + uj^\kappa$ is

$$ds^2 = (\bar{x}_t\cdot\bar{x}_t + 2u\bar{j}_t\cdot\bar{x}_t + u^2\bar{j}_t\cdot\bar{j}_t)\,dt^2 + 2\bar{j}\cdot\bar{x}_t\,dt\,du. \tag{44}$$

And *this form is negative definite.*
The Gaussian curvature is given by

$$K = \frac{\bar{j}_t\cdot\bar{j}_t}{(\bar{j}\cdot\bar{x}_t)^2} \qquad \text{if } \bar{j}\cdot\bar{x}_t \neq 0 \tag{45}$$

and is in general not zero.
Thus, *A ruled surface in R_4 with isotropic generators is not necessarily developable. If the surface is developable the ds^2 is a perfect square and the directrix is a minimal curve.*[28] *The surface is also minimal in the sense of least area.*

For classifications of isotropic manifolds we refer the reader to papers by J. Lense and M. Pinl and the Schouten and Struik II.[29, 30]

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THE INDUCTANCE OF TUBULAR CONDUCTORS OF ECCENTRIC-ANNULAR CROSS-SECTION[1]

By Thomas James Higgins[2]

1. Introduction

In Section 2 of this article we solve the following problem: to determine the inductance per unit length of two parallel cylindrical conductors of permeabilities μ_2 and μ_4, their cross-sections being eccentric annuli, the permeabilities of the media internal and external to them being, respectively, μ_1, μ_3, and μ_5. Now, while this general problem is of interest in itself, certain special cases of the same are even more so, this by virtue of their past history and present engineering significance.

Fifty-five years ago Lord Rayleigh[3] wrote:

"In §685 Maxwell takes the problem of two cylindrical conductors, the first of which conveys the outgoing and the second the (numerically equal) return current. The external radii are[4] r_2 and r_4; the internal radii r_1 and r_5; D the distance between centers. A possible difference in the magnetic quality is contemplated, the permeabilities for the material composing the cylinders being denoted by μ_2, μ_4, and that of the intervening space by μ_3.

The first correction I have to note relates merely to a slip of the pen. The result (22) should run

$$\frac{L}{l} = 2\mu_3 \log \frac{D^2}{r_2 r_4} + \tfrac{1}{2}\mu_2\left[\frac{r_2^2 - 3r_1^2}{r_2^2 - r_1^2} + \frac{4r_1^4}{(r_2^2 - r_1^2)^2}\log\frac{r_2}{r_1}\right] + \tfrac{1}{2}\mu_4\left[\frac{r_4^2 - 3r_5^2}{r_4^2 - r_5^2} + \frac{4r_5^4}{(r_4^2 - r_5^2)^2}\log\frac{r_4}{r_5}\right].$$

As printed in Maxwell's book the square brackets are omitted. This error does not affect the following formula (23), in which the cylinders are supposed to be solid. By putting r_1, r_5 equal to zero, we get

$$\frac{L}{l} = 2\mu_3 \log \frac{D^2}{r_2 r_4} + \tfrac{1}{2}(\mu_2 + \mu_4).$$

It must, however, be remarked that in the derivation of (22) Maxwell appears to have overlooked the effect of the matter composing one conductor in disturbing the lines of induction due to the current in the other. On this account the formula is correct only when the permeabilities μ are all equal, and the results

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[3] Rayleigh, *On the self-induction and resistance of straight conductors*, Philosophical Magazine, Vol. 21 (1886), pp. 381-394. See p. 382.

[4] The notation has been changed to conform with ours.

cannot be applied to iron wires without reservation. It would seem, however, that the error is of small importance when the wires are distant. The application to wires in contact, contemplated in §688, will hold good only for the non-magnetic metals."

Eight years later (in 1893), in a paper presented to the Cambridge Philosophical Society, MacDonald[5] gave a rigorous solution of Maxwell's problem for solid conductors. An abstract of MacDonald's paper is to be found in the *Proceedings* of the Society. Unhappily, therein the details of the solution are omitted; the problem is stated, the formula for the inductance is then given, and the remainder of the paper is devoted to numerical analyses of certain special cases of the general problem. These analyses comprise two tables showing the variation with spacing of the inductance per unit length of a line and the mutual mechanical forces exerted between the individual conductors constituting the line, one of these conductors being copper, the other iron of permeability 100, the two immersed in air.

Maxwell's problem for tubular conductors is but a special case of the general problem considered in Section 2. It is considered in detail in Section 3. No solution has hitherto been given.

With reference to the foregoing remarks, we briefly summarize the content of this article; the solution of a general problem of considerable scope is effected; certain special cases affording solutions to some long unsolved problems in electromagnetic theory are considered in detail; a numerical analysis of one of these cases is made, the same providing insight to a transmission line design of current interest; errors in two formulas given in MacDonald's paper are pointed out, and the right formulas given.

2. The formal solution

With reference to Figure 1, we seek A which satisfies the differential equations

$$\nabla^2 A_1 = 0 \qquad \text{over } C_1, \tag{1}$$

$$\nabla^2 A_2 = -4\pi\mu_2 w_2 \qquad \text{over } C_2, \tag{2}$$

$$\nabla^2 A_3 = 0 \qquad \text{over } C_3, \tag{3}$$

$$\nabla^2 A_4 = -4\pi\mu_4 w_4 \qquad \text{over } C_4, \tag{4}$$

$$\nabla^2 A_5 = 0 \qquad \text{over } C_5, \tag{5}$$

and the boundary conditions

$$A_1 = A_2 \quad \text{and} \quad \frac{1}{\mu_1}\frac{\partial A_1}{\partial n} = \frac{1}{\mu_2}\frac{\partial A_2}{\partial n} \quad \text{on the perimeter of } C_1. \tag{6}$$

$$A_2 = A_3 \quad \text{and} \quad \frac{1}{\mu_2}\frac{\partial A_2}{\partial n} = \frac{1}{\mu_3}\frac{\partial A_3}{\partial n} \quad \text{on the perimeter of } C_2. \tag{7}$$

[5] H. M. MacDonald, *The self-induction of two parallel conductors*, Proceedings of the Cambridge Philosophical Society, Vol. 7 (1889–1892), pp. 259–261.

(8) $A_3 = A_4$ and $\frac{1}{\mu_3}\frac{\partial A_3}{\partial n} = \frac{1}{\mu_4}\frac{\partial A_4}{\partial n}$ on the perimeter of C_3.

(9) $A_4 = A_5$ and $\frac{1}{\mu_4}\frac{\partial A_4}{\partial n} = \frac{1}{\mu_5}\frac{\partial A_5}{\partial n}$ on the perimeter of C_4.

(10) $A_5 =$ a constant at infinity.

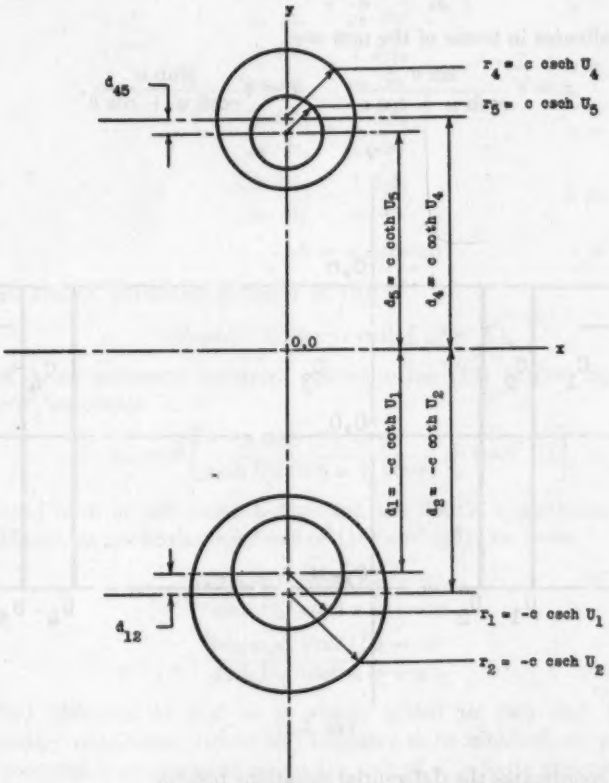

FIG. 1.

The boundary conditions are obtained as follows: (6) to (9) inclusive from knowledge that the vector potential and the tangential component of magnetic intensity must be continuous at each point in the interface between two media of different permeability; (10) from knowledge that the vector potential vanishes at infinity except for an arbitrary constant.

To determine the vector potential, we find particular solutions of (2) and (4), and add to them solutions of $\nabla^2 A = 0$ such that the boundary conditions

are satisfied. To facilitate this task, we make a change of variables. The conformal transformation

$$f(z) = -i \log \frac{-z + ic}{z + ic} \tag{11}$$

maps the (x, y) plane on the (u, v) plane as shown in Figure 2. The scale factor is

$$\left| \frac{df(z)}{dz} \right| = \frac{1}{c} (\cosh u + \cos v); \tag{12}$$

the old coordinates in terms of the new are

$$x = c \frac{\sin v}{\cosh u + \cos v}, \qquad y = c \frac{\sinh u}{\cosh u + \cos v}. \tag{13}$$

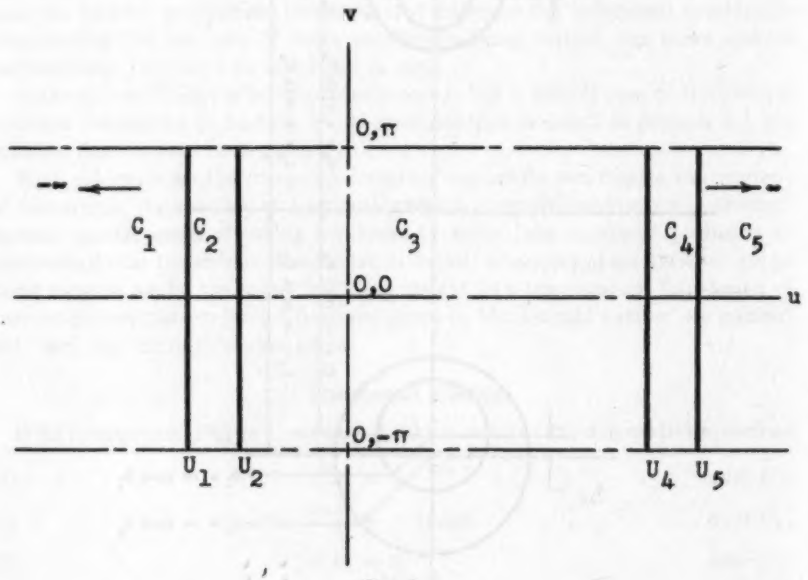

FIG. 2.

In the new coordinates the differential equations become

$$\nabla^2 A_1 = 0 \qquad -\infty \leq u \leq U_1, \tag{14}$$

$$\nabla^2 A_2 = -\frac{4\pi\mu_2 w_2 c^2}{(\cosh u + \cos v)^2} \qquad U_1 \leq u \leq U_2, \tag{15}$$

$$\nabla^2 A_3 = 0 \qquad U_2 \leq u \leq U_4, \tag{16}$$

$$\nabla^2 A_4 = -\frac{4\pi\mu_4 w_4 c^2}{(\cosh u + \cos v)_2} \qquad U_4 \leq u \leq U_5, \tag{17}$$

$$\nabla^2 A_5 = 0 \qquad U_5 \leq u \leq \infty. \tag{18}$$

The boundary conditions yield

$$A_1 = A_2 \qquad u = U_1, \tag{19}$$

$$A_2 = A_3 \qquad u = U_2, \tag{20}$$

$$A_3 = A_4 \qquad u = U_4, \tag{21}$$

$$A_5 = A_4 \qquad u = U_5, \tag{22}$$

$$\frac{1}{\mu_1}\frac{\partial A_1}{\partial n} = \frac{1}{\mu_2}\frac{\partial A_2}{\partial n} \qquad u = U_1, \tag{23}$$

$$\frac{1}{\mu_3}\frac{\partial A_3}{\partial n} = \frac{1}{\mu_2}\frac{\partial A_3}{\partial n} \qquad u = U_2, \tag{24}$$

$$\frac{1}{\mu_3}\frac{\partial A_3}{\partial n} = \frac{1}{\mu_4}\frac{\partial A_4}{\partial n} \qquad u = U_4, \tag{25}$$

$$\frac{1}{\mu_5}\frac{\partial A_5}{\partial n} = \frac{1}{\mu_4}\frac{\partial A_4}{\partial n} \qquad u = U_5, \tag{26}$$

$$A = \text{a constant} \qquad u = 0, v = \pm\pi. \tag{27}$$

A well known particular solution of (2) is

$$-\pi\mu_2 w_2[x^2 + (y - c\coth U_2)^2 + K], \tag{28}$$

where K is an arbitrary constant. Substituting (13) in (28) and taking K equal to c^2, we obtain

$$-\pi\mu_2 w_2 c^2\left[\frac{2\sinh(U_2 - u)}{\sinh U_2(\cosh u + \cos v)} + \coth^2 U_2\right]. \tag{29}$$

The second term in (29) being a constant, the first is a particular solution of (15). Hence, as particular solutions of (15) and (17), we have

$$-\frac{2\pi\mu_2 w_2 c^2}{\sinh U_2}\frac{\sinh(U_2 - u)}{\cosh u + \cos v}. \tag{30}$$

$$-\frac{2\pi\mu_4 w_4 c^2}{\sinh U_4}\frac{\sinh(U_4 - u)}{\cosh u + \cos v}. \tag{31}$$

To find solutions of $\nabla^2 A = 0$ which, added to (30) and (31), enable the boundary conditions (19) to (27) inclusive to be satisfied, we note that the vector potential is symmetrical about the axis of u, is finite everywhere, and is equal to a constant at infinity. These observations lead to the following expressions:

$$A_1 = \sum_{n=0}^{\infty} L_n \epsilon^{nu}\cos nv, \tag{32}$$

$$A_2 = \frac{2\pi\mu_2 w_2 c^2}{\sinh U_2}\frac{\sinh(u - U_2)}{\cosh u + \cos v} + 0u + \sum_{n=0}^{\infty}(M_n\epsilon^{nu} + N_n\epsilon^{-nu})\cos nv, \tag{33}$$

$$A_3 = Tu + \sum_{n=0}^{\infty} (R_n \epsilon^{nu} + S_n \epsilon^{-nu}) \cos nv, \tag{34}$$

$$A_4 = \frac{2\pi\mu_4 w_4 c^2}{\sinh U_4} \frac{\sinh (u - U_4)}{\cosh u + \cos v} + Wu + \sum_{n=0}^{\infty} (X_n \epsilon^{nu} + Y_n \epsilon^{-nu}) \cos nv, \tag{35}$$

$$A_5 = \sum_{n=0}^{\infty} Z_n \epsilon^{-nu} \cos nv. \tag{36}$$

To obtain the unknown coefficients, we apply the boundary conditions. Recalling[6] that

$$\begin{aligned} (\cosh u + \cos v)^{-1} &= -\frac{1}{\sinh u}\left[1 + 2\sum_{n=1}^{\infty} (-1)^n \epsilon^{nu} \cos nv\right] \qquad u \leq 0 \\ &= \frac{1}{\sinh u}\left[1 + 2\sum_{n=1}^{\infty} (-1)^n \epsilon^{-nu} \cos nv\right] \qquad u \geq 0, \end{aligned} \tag{37}$$

we obtain from (19) to (26) inclusive

$$\begin{aligned} &\sum_{n=0}^{\infty} L_n \epsilon^{nU_1} \cos nv \\ &= -\frac{2\pi\mu_2 w_2 c^2 \sinh (U_1 - U_2)}{\sinh U_1 \sinh U_2}\left[1 + 2\sum_{n=1}^{\infty} (-1)^n \epsilon^{nU_1} \cos nv\right] + 0U_1 \\ &\quad + \sum_{n=0}^{\infty} (M_n \epsilon^{nU_1} + N_n \epsilon^{-nU_1}) \cos nv, \end{aligned} \tag{38}$$

$$\begin{aligned} &TU_2 + \sum_{n=0}^{\infty} (R_n \epsilon^{nU_2} + S_n \epsilon^{-nU_2}) \cos nv \\ &\quad = 0U_2 + \sum_{n=0}^{\infty} (M_n \epsilon^{nU_2} + N_n \epsilon^{-nU_2}), \end{aligned} \tag{39}$$

$$\begin{aligned} &TU_4 + \sum_{n=0}^{\infty} (R_n \epsilon^{nU_4} + S_n \epsilon^{-nU_4}) \cos nv \\ &\quad = WU_4 + \sum_{n=0}^{\infty} (X_n \epsilon^{nU_4} + Y_n \epsilon^{-nU_4}), \end{aligned} \tag{40}$$

$$\begin{aligned} &\sum_{n=0}^{\infty} Z_n \epsilon^{-nU_5} \cos nv \\ &= \frac{2\pi\mu_4 w_4 c^2 \sinh (U_5 - U_4)}{\sinh U_4 \sinh U_5}\left[1 + 2\sum_{n=1}^{\infty} (-1)^n \epsilon^{-nU_5} \cos nv\right] + WU_5 \\ &\quad + \sum_{n=0}^{\infty} (X_n \epsilon^{nU_5} + Y_n \epsilon^{-nU_5}) \cos nv, \end{aligned} \tag{41}$$

[6] This expansion is but the Fourier series, derived as usual.

$$b_n = \frac{2}{\pi}\int_0^{\pi} \frac{\cos nv}{\cosh u + \cos v}\, dv = -\frac{2(-1)^n \epsilon^{nu}}{\sinh u}, \qquad u < 0.$$

$$\frac{1}{\mu_1}\sum_{n=0}^{\infty} L_n n\epsilon^{nU_1}\cos nv = \frac{1}{\mu_2}\left\{-\frac{2\pi\mu_2 w_2 c^2}{\sinh^2 U_1}\left[1+2\sum_{n=1}^{\infty}(-1)^n\epsilon^{nU_1}\cos nv\right]\right.$$

$$-\frac{4\pi\mu_2 w_2 c^2\sinh(U_1-U_2)}{\sinh U_1\sinh U_2}\sum_{n=1}^{\infty} n(-1)^n\epsilon^{nU_1}\cos nv \tag{42}$$

$$\left.+\sum_{n=0}^{\infty} n(M_n\epsilon^{nU_1}-N_n\epsilon^{-nU_1})\cos nv+0\right\},$$

$$\frac{1}{\mu_3}\left\{T+\sum_{n=0}^{\infty} n(R_n\epsilon^{nU_2}-S_n\epsilon^{-nU_2})\cos nv\right\}$$

$$=\frac{1}{\mu_2}\left\{-\frac{2\pi\mu_2 w_2 c^2}{\sinh^2 U_2}\left[1+2\sum_{n=1}^{\infty}(-1)^n\epsilon^{nU_2}\cos nv\right]\right. \tag{43}$$

$$\left.+\sum_{n=0}^{\infty} n(M_n\epsilon^{nU_2}-N_n\epsilon^{-nU_2})\cos nv+0\right\},$$

$$\frac{1}{\mu_3}\left\{T+\sum_{n=0}^{\infty} n(R_n\epsilon^{nU_4}-S_n\epsilon^{-nU_4})\cos nv\right.$$

$$=\frac{1}{\mu_4}\left\{\frac{2\pi\mu_4 w_4 c^2}{\sinh^2 U_4}\left[1+2\sum_{n=0}^{2\infty}(-1)^n\epsilon^{-nU_4}\cos nv\right]\right. \tag{44}$$

$$\left.\left.+\sum_{n=0}^{\infty} n(X_n\epsilon^{nU_4}-Y_n\epsilon^{-nU_4})\cos nv+W\right\},\right.$$

$$\frac{1}{\mu_5}\sum_{n=0}^{\infty}-Z_n n\epsilon^{-nU_5}\cos nv=\frac{1}{\mu_4}\left\{\frac{2\pi\mu_4 w_4 c^2}{\sinh^2 U_5}\left[1+2\sum_{n=1}^{\infty}(-1)^n\epsilon^{-nU_5}\cos nv\right]\right.$$

$$-\frac{4\pi\mu_4 w_4 c^2\sinh(U_5-U_4)}{\sinh U_4\sinh U_5}\sum_{n=1}^{\infty} n(-1)^n\epsilon^{-nU_5}\cos nv \tag{45}$$

$$\left.+\sum_{n=0}^{\infty} n(X_n\epsilon^{nU_5}-Y_n\epsilon^{-nU_5})\cos nv+W\right\}.$$

Equating in each equation the coefficients of the like values of cos nv and utilizing the identities

$$r_1=-c\operatorname{csch}U_1 \qquad d_1=-c\coth U_1$$

$$r_2=-c\operatorname{csch}U_2 \qquad d_2=-c\coth U_2$$

$$r_4=\ \ c\operatorname{csch}U_4 \qquad d_4=\ \ c\coth U_4$$

$$r_5=\ \ c\operatorname{csch}U_5 \qquad d_5=\ \ c\coth U_5$$

$$d_{12}=-c\frac{\sinh(U_1-U_2)}{\sinh U_1\sinh U_2}, \qquad d_{45}=c\frac{\sinh(U_5-U_4)}{\sinh U_4\sinh U_5},$$

we obtain from (38) to (45) inclusive

$$L_0=2\pi\mu_2 w_2 d_{12}c+(M_0+N_0)+0U_1, \tag{46}$$

$$(47)\quad L_n = 4\pi\mu_3 w_2 d_{12} c(-1)^n + M_n + N_n \epsilon^{-2U_1},$$

$$(48)\quad TU_2 + (R_0 + S_0) = 0U_2 + (M_0 + N_0),$$

$$(49)\quad R_n + S_n \epsilon^{-2nU_2} = M_n + N_n \epsilon^{-2nU_2},$$

$$(50)\quad TU_4 + (R_0 + S_0) = WU_4 + (X_0 + Y_0),$$

$$(51)\quad R_n + S_n \epsilon^{-2nU_4} = X_n + Y_n \epsilon^{-2nU_4},$$

$$(52)\quad Z_0 = 2\pi\mu_4 w_6 d_{45} c + (X_0 + Y_0) + WU_5,$$

$$(53)\quad Z_n = 4\pi\mu_4 w_6 d_{45} c(-1)^n + X_n \epsilon^{2nU_5} + Y_n,$$

$$(54)\quad 0 = 2\pi\mu_3 w_2 r_1^2,$$

$$(55)\quad \frac{n}{\mu_1} L_n = -4\pi w_2 r_1^2 (-1)^n + 4\pi w_2 d_{12} c(-1)^n n + \frac{n}{\mu_2}(M_n - N_n \epsilon^{-2nU_1}),$$

$$(56)\quad \frac{1}{\mu_3} T = -2\pi w_2 r_2^2 + \frac{1}{\mu_2} 0,$$

$$(57)\quad \frac{n}{\mu_3}(R_n - S_n \epsilon^{-2nU_2}) = -4\pi w_2 r_2^2 (-1)^n + \frac{n}{\mu_2}(M_n - N_n \epsilon^{-2nU_2}),$$

$$(58)\quad \frac{1}{\mu_3} T = 2\pi w_4 r_4^2 \frac{1}{\mu_4} W,$$

$$(59)\quad \frac{n}{\mu_3}(R_n \epsilon^{2nU_4} - S_n) = 4\pi w_4 r_4^2 (-1)^n + \frac{n}{\mu_4}(X_n \epsilon^{2nU_4} - Y_n),$$

$$(60)\quad W = -2\pi\mu_4 w_4 r_5^2,$$

$$(61)\quad \frac{n}{\mu_5} Z_n = -4\pi w_4 r_5^2 (-1)^n + 4\pi w_4 d_{45} c(-1)^n n - \frac{n}{\mu_4}(X_n \epsilon^{2nU_5} - Y_n).$$

Although we thus obtain sixteen equations, but fifteen are linearly independent. This, however, is to be expected; for (7) is equivalent to stating that the expression for the vector potential contains an arbitrary constant. Accordingly, taking $(R_0 + S_0)$ as this arbitrary constant, we can solve the foregoing set of equations for the remaining fifteen unknown: $(R_0 + S_0)$, $(M_0 + N_0)$, and $(X_0 + Y_0)$ are considered as three unknown, the extra letters having been introduced for the sake of symmetry when formulating (33), (34), and (35).

The set of eight even-numbered equations can be solved without difficulty for seven of the unknowns, an eighth appearing as a parameter in the solutions:

$$(62)\quad L_0 = 2\pi\mu_2 w_2 d_{12} c + 2\pi\mu_2 w_2 r_1^2 (U_1 - U_2) - 2\pi\mu_3 w_2 (r_2^2 - r_1^2) U_2 + (R_0 + S_0),$$

$$(63)\quad (M_0 + N_0) = -2\pi\mu_2 w_2 r_1^2 U_2 - 2\pi\mu_3 w_2 (r_2^2 - r_1^2) U_2 + (R_0 + S_0),$$

$$(64)\quad 0 = 2\pi\mu_2 w_2 r_1^2,$$

$$T = -2\pi\mu_3 w_2(r_2^2 - r_1^2) = 2\pi\mu_3 w_4(r_4^2 - r_5^2), \tag{65}$$

$$W = -2\pi\mu_4 w_4 r_5^2, \tag{66}$$

$$(X_0 + Y_0) = 2\pi\mu_4 w_4 r_5^2 U_4 + 2\pi\mu_3 w_4(r_4^2 - r_5^2)U_4 + (R_0 + S_0), \tag{67}$$

$$Z_0 = 2\pi\mu_4 w_4 d_{45} c + 2\pi\mu_4 w_4 r_5^2(U_4 - U_5) + 2\pi\mu_3 w_4(r_4^2 - r_5^2)U_4 + (R_0 + S_0). \tag{68}$$

We shall not solve the set of odd-numbered equations for the remaining eight unknown coefficients. To do so would require very heavy algebra, and the resulting general expressions for the coefficients would be hopelessly unwieldy. Rather, for a particular problem, it is far better to insert in these equations the given values of r, d, and μ and to solve the set of equations thus obtained. This latter is usually much simpler than the original set: the symmetry associated with it enables one to use the theory of diagonally-reflected symmetrical determinants to the end that algebraical and numerical labor is greatly reduced. While, in general, this procedure requires the minimum of labor, there are some problems (for example, all inductance problems) wherein only the coefficients M_n, N_n, X_n, and Y_n are required. For such problems it is convenient to have at hand the set of four equations to be obtained by eliminating L_n, R_n, S_n, and Z_n between the original set of eight.

Accordingly, carrying out the necessary analysis, we obtain

$$M_n\left(\frac{\mu_2}{\mu_2} - 1\right)\epsilon^{2nU_2} - N_n\left(\frac{\mu_2}{\mu_2} + 1\right) - X_n\left(\frac{\mu_2}{\mu_4} - 1\right)\epsilon^{2nU_4} + Y_n\left(\frac{\mu_2}{\mu_4} + 1\right) = k_1, \tag{69}$$

$$- X_n\left(\frac{\mu_5}{\mu_4} + 1\right) + Y_n\left(\frac{\mu_5}{\mu_4} - 1\right)\epsilon^{-2nU_5} = k_2, \tag{70}$$

$$M_n\left(\frac{\mu_1}{\mu_2} - 1\right)\epsilon^{2nU_1} - N_n\left(\frac{\mu_1}{\mu_2} + 1\right) = k_3, \tag{71}$$

$$M_n\left(\frac{\mu_3}{\mu_2} + 1\right) - N_n\left(\frac{\mu_3}{\mu_2} - 1\right)\epsilon^{-2nU_3} - X_n\left(\frac{\mu_3}{\mu_4} + 1\right) + Y_n\left(\frac{\mu_3}{\mu_4} - 1\right)\epsilon^{-2nU_4} = k_4, \tag{72}$$

wherein

$$k_1 = 4\pi\mu_3 \frac{(-1)^n}{n}[w_2 r_2^2 \epsilon^{2nU_2} + w_4 r_4^2], \tag{73}$$

$$k_2 = -4\pi w_4 \frac{(-1)^n}{n}[cnd_{45}(\mu_5 - \mu_4) - \mu_5 r_5^2]\epsilon^{-2nU_5}, \tag{74}$$

$$k_3 = -4\pi w_2 \frac{(-1)^n}{n}[cnd_{12}(\mu_1 - \mu_2) - \mu_1 r_1^2]\epsilon^{2nU_1}, \tag{75}$$

$$k_4 = 4\pi\mu_3 \frac{(-1)^n}{n}[w_2 r_2^2 + w_4 r_4^2 \epsilon^{-2nU_4}]. \tag{76}$$

We now proceed to determine L', the inductance per unit length of line, from the condition that W', the energy stored in the electromagnetic field per unit length of line, equals $\frac{1}{2}L'I^2$. From well known theory we have

$$
\begin{aligned}
W' &= \tfrac{1}{2}\int\int_{C_2} A_2 w_2 \, dv \, du + \tfrac{1}{2}\int\int_{C_4} A_4 w_4 \, dv \, du \\
&= \frac{w_2 c^2}{2}\int_{U_1}^{U_2}\int_{-\pi}^{\pi} \frac{A_2 \, dv \, du}{(\cosh u + \cos v)^2} \\
&+ \frac{w_4 c^2}{2}\int_{U_4}^{U_5}\int_{-\pi}^{\pi} \frac{A_4 \, dv \, du}{(\cosh u + \cos v)^2} \qquad (77) \\
&= w_2 c^2 \int_{U_1}^{U_2}\int_{0}^{\pi} \frac{A_2 \, dv \, du}{(\cosh u + \cos v)^2} \\
&+ w_4 c^2 \int_{U_4}^{U_5}\int_{0}^{\pi} \frac{A_4 \, dv \, du}{(\cosh u + \cos v)^2}.
\end{aligned}
$$

To evaluate W' we require the following integrals[7]:

$$
c^2 \int_{U_1}^{U_2}\int_{0}^{\pi} \frac{1}{(\cosh u + \cos v)^2} \, dv \, du = \frac{\pi}{2}(r_2^2 - r_1^2), \qquad (78)
$$

$$
c^2 \int_{U_4}^{U_5}\int_{0}^{\pi} \frac{1}{(\cosh u + \cos v)^2} \, dv \, du = \frac{\pi}{2}(r_4^2 - r_5^2) \qquad (79)
$$

$$
\begin{aligned}
&c^4 \int_{U_1}^{U_2}\int_{0}^{\pi} \frac{\sinh (u - U_2)}{\sinh U_2 (\cosh u + \cos v)^3} \, dv \, du \\
&\qquad = -\frac{\pi}{2}(d_1 d_2 r_1^2 - r_1^2 c^2 - \tfrac{3}{4}r_1^4 - \tfrac{1}{4}r_2^4), \qquad (80)
\end{aligned}
$$

$$
\begin{aligned}
&c^4 \int_{U_4}^{U_5}\int_{0}^{\pi} \frac{\sinh (u - U_4)}{\sinh U_4 (\cosh u + \cos v)^3} \, dv \, du \\
&\qquad = -\frac{\pi}{2}(d_4 d_5 r_5^2 - c^2 r_5^2 - \tfrac{3}{4}r_5^4 - \tfrac{1}{4}r_4^4), \qquad (81)
\end{aligned}
$$

$$
\begin{aligned}
&c^2 \int_{U_1}^{U_2}\int_{0}^{\pi} \frac{u}{(\cosh u + \cos v)^2} \, dv \, du \\
&\qquad = \frac{\pi}{2}[r_2^2 U_2 - r_1^2 U_1 + c(d_2 - d_1)], \qquad (82)
\end{aligned}
$$

$$
\begin{aligned}
&c^2 \int_{U_4}^{U_5}\int_{0}^{\pi} \frac{u}{(\cosh u + \cos v)^2} \, dv \, du \\
&\qquad = \frac{\pi}{2}[r_4^2 U_4 - r_5^2 U_5 + c(d_4 - d_5)], \qquad (83)
\end{aligned}
$$

[7] The values of the inner integrals are to be found in D. Bierens de Haan's well-known tables. The outer integrals can then be evaluated by direct integration. We omit details as elementary and space-consuming.

$$c^2 \int_{U_1}^{U_2} \int_0^{\pi} \frac{\epsilon^{nu} \cos nv}{(\cosh u + \cos v)^2}\, dv\, du = \frac{\pi(-1)^n}{2} (r_2^2 \epsilon^{2nU_2} - r_1^2 \epsilon^{2nU_1}), \tag{84}$$

$$c^2 \int_{U_4}^{U_5} \int_0^{\pi} \frac{\epsilon^{nu} \cos nv}{(\cosh u + \cos v)^2}\, dv\, du = \frac{\pi(-1)^n}{2} [r_4^2 - r_5^2 + 2nc(d_4 - d_5)], \tag{85}$$

$$c^2 \int_{U_1}^{U_2} \int_0^{\pi} \frac{\epsilon^{-nu} \cos nv}{(\cosh u + \cos v)^2}\, dv\, du = \frac{\pi(-1)^n}{2} [r_2^2 - r_1^2 + 2nc(d_2 - d_1)], \tag{86}$$

$$c^2 \int_{U_4}^{U_5} \int_0^{\pi} \frac{\epsilon^{-nu} \cos nv}{(\cosh u + \cos v)^2}\, dv\, du = \frac{\pi(-1)^n}{2} [(r_4^2 \epsilon^{-2nU_4} - r_5^2 \epsilon^{-2nU_5})]. \tag{87}$$

Substituting (33) and (35) in (77), and carrying out the required integrations by aid of (78) to (87) inclusive, we obtain

$$\begin{aligned} 2W' = &- 2\pi^2 \mu_2 w_2^2 (d_1 d_2 r_1^2 - c^2 r_1^2 - \tfrac{3}{4} r_1^4 - \tfrac{1}{4} r_2^4) \\ &- 2\pi^2 \mu_4 w_4^2 (d_4 d_5 r_5^2 - c^2 r_5^2 - \tfrac{3}{4} r_5^4 - \tfrac{1}{4} r_4^4) \\ &+ \pi w_2 (r_2^2 - r_1^2)(M_0 + N_0) \\ &+ \pi w_4 (r_4^2 - r_5^2)(X_0 + Y_0) \\ &+ \pi w_2 [r_2^2 U_2 - r_1^2 U_1 + c(d_2 - d_1)]0 \\ &+ \pi w_4 [r_4^2 U_4 - r_5^2 U_5 + c(d_4 - d_5)]W \\ &+ \pi w_2 \sum_{n=1}^{\infty} (-1)^n (r_2^2 \epsilon^{2nU_2} - r_1^2 \epsilon^{2nU_1}) M_n \\ &+ \pi w_4 \sum_{n=1}^{\infty} (-1)^n [r_4^2 \epsilon^{-2nU_4} - r_5^2 \epsilon^{-2nU_5}] X_n \\ &+ \pi w_2 \sum_{n=1}^{\infty} (-1)^n [r_2^2 - r_1^2 + 2nc(d_2 - d_1)] N_n \\ &+ \pi w_4 \sum_{n=1}^{\infty} (-1)^n [r_4^2 - r_5^2 + 2nc(d_4 - d_5)] X_n . \end{aligned} \tag{88}$$

Substituting for $(M_0 + N_0)$, $(X_0 + Y_0)$, 0, and W from (63), (64), (66), and (67), we have after collecting terms

$$
\begin{aligned}
2W' = {} & 2\pi^2 \mu_2 w_2^2 [\tfrac{3}{4} r_1^4 + \tfrac{1}{4} r_2^4 - d_1 d_2 r_1^2 + c^2 r_1^2] \\
& + 2\pi^2 \mu_4 w_4^2 [\tfrac{3}{4} r_5^4 + \tfrac{1}{4} r_4^4 - d_4 d_5 r_5^2 + c^2 r_5^2] \\
& - 2\pi^2 w_2^2 (r_2^2 - r_1^2)[\mu_2 r_1^2 U_2 + \mu_3 (r_2^2 - r_1^2) U_2] \\
& + 2\pi^2 w_4^2 (r_4^2 - r_5^2)[\mu_4 r_5^2 U_4 + \mu_3 (r_4^2 - r_5^2) U_4] \\
& + 2\pi^2 \mu_2 w_2^2 r_1^2 [r_2^2 U_2 - r_1^2 U_1 + c(d_2 - d_1)] \\
& - 2\pi^2 \mu_4 w_4^2 r_5^2 [r_4^2 U_4 - r_5^2 U_5 + c(d_4 - d_5)] \\
& + \pi w_2 \sum_{n=1}^{\infty} (-1)^n [r_2^2 \epsilon^{2nU_2} - r_1^2 \epsilon^{2nU_1}] M_n \\
& + \pi w_4 \sum_{n=1}^{\infty} (-1)^n [r_4^2 \epsilon^{-2nU_4} - r_5^2 \epsilon^{-2nU_5}] Y^n \\
& + \pi w_2 \sum_{n=1}^{\infty} (-1)^n [r_2^2 - r_1^2 + 2nc(d_2 - d_1)] N_n \\
& + \pi w_4 \sum_{n=1}^{\infty} (-1)^n [r_4^2 - r_5^2 + 2nc(d_4 - d_5)] X_n .
\end{aligned}
\tag{89}
$$

Finally, L' can be obtained from

$$
L' = 2W'/I^2 \text{ abhenries per centimeter of line.} \tag{90}
$$

Thus, our problem in its most general aspect is solved. When one considers that the permeability of each of the five media and the radius and relative position of each of the four circular boundaries can be varied at will, it is readily understood that the solution of an exceedingly large number of special problems can be obtained from (90) by appropriately assigning values to the various parameters. We shall now consider some cases of especial engineering interest.

3. Special cases

I. We have

$$
\mu_1 = \mu_5 , \qquad \mu_2 = \mu_3 = \mu_4 , \qquad U_1 = -U_5 , \qquad U_2 = -U_4 ,
$$

and

$$
d_{12} = d_{45} = 0.
$$

Substituting these values for μ, U, and d in (89), we have

$$
\begin{aligned}
2W' = {} & \pi^2 \mu_2 w_2^2 (3r_1^4 + r_2^4 - 4 r_1^2 r_2^2) \\
& - 4\pi^2 \mu_2 w_2^2 (r_2^2 - r_1^2) U_2 r_2^2 \\
& + 4\pi^2 \mu_2 w_2^2 r_1^2 (r_2^2 U_2 - r_1^2 U_1) \\
& + \pi w_2 \sum_{n=1}^{\infty} (-1)^n (r_2^2 \epsilon^{2nU_2} - r_1^2 \epsilon^{2nU_1})(M_n - Y_n) \\
& + \pi w_2 \sum_{n=1}^{\infty} (-1)^n (r_2^2 - r_1^2)(N_n - X_n).
\end{aligned}
\tag{91}
$$

To determine the unknown coefficients in (91), we substitute the given values of μ, U, and d in (69) to (72) inclusive. Thus,

$$-2N_n + 2Y_n = k_1, \tag{92}$$

$$-\left(\frac{\mu_1}{\mu_2}+1\right)X_n + \left(\frac{\mu_1}{\mu_2}-1\right)Y_n\epsilon^{2nU_1} = k_2, \tag{93}$$

$$\left(\frac{\mu_1}{\mu_2}-1\right)M_n\epsilon^{2nU_1} - \left(\frac{\mu_1}{\mu_2}+1\right)N_n = k_3, \tag{94}$$

$$2M_n - 2X_n = k_4, \tag{95}$$

where

$$k_1 = -k_4 = 4\pi\mu_2 w_2 r_2^2 \frac{(-1)^n}{n}(\epsilon^{2nU_2} - 1), \tag{96}$$

$$k_2 = -k_3 = -4\pi\mu_1 w_2 r_1^2 \frac{(-1)^n}{n}\epsilon^{2nU_1}. \tag{97}$$

For convenience, we rewrite this set of equations as follows:

$$-N_n + Y_n = K', \tag{98}$$

$$-X_n + PY_n\epsilon^{2nU_1} = K'', \tag{99}$$

$$PM_n\epsilon^{2nU_1} - N_n = -K'', \tag{100}$$

$$M_n - X_n = -K', \tag{101}$$

where

$$P = \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2}, \tag{102}$$

$$K' = 2\pi\mu_2 w_2 r_2^2 \frac{(-1)^n}{n}(\epsilon^{2nU_2} - 1), \tag{103}$$

$$K'' = -2\pi\mu_2 w_2 r_1^2(P+1)\frac{(-1)^n}{n}\epsilon^{2nU_1}. \tag{104}$$

Solving this set of equations, we obtain

$$M_n = -Y_n = -\frac{K' + K''}{1 + P\epsilon^{2nU_1}}, \tag{105}$$

$$N_n = -X_n = \frac{K'' - K'P\epsilon^{2nU_1}}{1 + P\epsilon^{2nU_1}}. \tag{106}$$

Now, for $P\epsilon^{2nu} < 1$, we have by the binomial theorem

$$(1 + P\epsilon^{2nu})^{-1} = 1 + \sum_{m=1}^{\infty}(-P\epsilon^{2nu})^m. \tag{107}$$

Accordingly, substituting (107) in (105) and (106), we have

$$M_n = -(K' + K'')\left(1 + \sum_{m=1}^{\infty}[-P\epsilon^{2nU_1}]^m\right). \tag{108}$$

$$N_n = (K'' - K'P\epsilon^{2nU_1})\left(1 + \sum_{m=1}^{\infty}[-P\epsilon^{2nU_1}]^m\right). \tag{109}$$

Substituting (108) and (109) in (91) yields

$$\begin{aligned} 2W' &= \pi^2\mu_2 w_2^2(3r_1^4 + r_2^4 4r_1^2r_2^2) - 4\pi^2\mu_2 w_2^2(r_2^2 - r_1^2)U_2 r_2^2 \\ &\quad + 4\pi^2\mu_2 w_2^2(r_2^2U_2 - r_1^2U_1)\, r_1^2 \\ &\quad - 2\pi w_2 \sum_{n=1}^{\infty}(-1)^n(r_2^2\epsilon^{2nU_2} - r_1^2\epsilon^{2nU_1})(K' + K'')\left(1 + \sum_{m=1}^{\infty}[-P\epsilon^{2nU_1}]^m\right) \\ &\quad + 2\pi w_2 \sum_{n=1}^{\infty}(-1)^n(r_2^2 - r_1^2)(K'' - K'P\epsilon^{2nU_1})\left(1 + \sum_{m=1}^{\infty}[-P\epsilon^{2nU_1}]^m\right). \end{aligned} \tag{110}$$

Substituting (103) and (104) in (110) and arranging the resulting expression as a series in powers of P, we have

$$\begin{aligned} 2W' &= \pi^2\mu_2 w_2^2(3r_1^4 + r_2^4 - 4r_1^2r_2^2) - 4\pi^2\mu_2 w_2^2(r_2^4U_2 - 2r_1^2r_2^2U_2 + r_1^4U_1) \\ &\quad - 4\pi^2\mu_2 w_2^2 \sum_{n=1}^{\infty}(r_2^4\epsilon^{4nU_2} - r_2^4\epsilon^{2nU_2} - 2r_1^2r_2^2\epsilon^{2n(U_1+U_2)} \\ &\quad + 2r_1^2r_2^2\epsilon^{2nU_1} + r_1^4\epsilon^{4nU_1} - r_1^4\epsilon^{2nU_1})n^{-1} \\ &\quad + 4\pi^2\mu_2 w_2^2 P \sum_{n=1}^{\infty}\epsilon^{2nU_1}[r_2^2(\epsilon^{2nU_2} - 1) - r_1^2(\epsilon^{2nU_1} - 1)^2]^2 n^{-1} \\ &\quad - 4\pi^2\mu_2 w_2^2 P^2 \sum_{n=1}^{\infty}\epsilon^{4nU_1}[r_2^2(\epsilon^{2nU_2} - 1) - r_1^2(\epsilon^{2nU_1} - 1)^2]^2 n^{-1} \\ &\quad - 4\pi^2\mu_2 w_2^2 \sum_{k=3}^{\infty}\sum_{n=1}^{\infty}[-P\epsilon^{2nU_1}]^k[r_2^2(\epsilon^{2nU_2} - 1) - r_1^2(\epsilon^{2nU_1} - 1)^2]^2 n^{-1}. \end{aligned} \tag{111}$$

Recalling that for $u < 0$

$$\sum_{n=1}^{\infty}\frac{\epsilon^{2nu}}{n} = -[u + \log\sinh(-u)], \tag{112}$$

we have from (112)

$$\begin{aligned} 2W' &= \pi^2\mu_2 w_2^2(3r_1^4 + r_2^4 - 4r_1^2r_2^2) \\ &\quad + 4\pi^2\mu_2 w_2^2\left(r_2^4\log\frac{D}{r_2} + r_1^4\log\frac{D}{r_1} - 2r_1^2r_2^2\log\frac{D}{r_2}\right) \\ &\quad + 4\pi^2\mu_2 w_2^2 P\left(r_2^4\log\frac{D^2}{D^2 - r_2^2} + r_1^4\log\frac{D^2}{D^2 - r_1^2} - 2r_1^2r_2^2\log\frac{r_2D^2}{r_1(D^2 - r_2^2)}\right) \\ &\quad + \cdots, \end{aligned} \tag{113}$$

where D is the distance between the centers of the conductors.

Finally, bearing in mind that

$$I^2 = \pi^2 w_2^2(r_2^2 - r_1^2)^2, \tag{114}$$

we have on dividing each member of (113) by the right-hand member of (114)

$$L = \mu_2\left[\frac{r_2^2 - 3r_1^2}{r_2^2 - r_1^2} + \frac{4r_1^4}{(r_2^2 - r_1^2)^2}\log\frac{r_2}{r_1} + 4\log\frac{D}{r_2}\right.$$

$$+ \frac{4P}{(r_2^2 - r_1^2)^2}\left(r_2^4\log\frac{D^2}{D^2 - r_2^2} + r_1^4\log\frac{D^2}{D^2 - r_1^2} - 2r_1^2r_2^2\log\frac{r_2D^2}{r_1(D^2 - r_2^2)}\right) \tag{115}$$

$$\left. + \cdots\right].$$

II. If the conditions are the same as in I except that $r_1 \neq r_5$ and $r_2 \neq r_4$, the solution can be obtained directly from (115) by consideration of the physics of the problem. The total inductance will be the sum of two expressions, one due to each conductor, identical except for the interchange of r_1 and r_2 with r_4 and r_5 respectively. Accordingly, from (115) we obtain by inspection

$$L = \tfrac{1}{2}\mu_2\left[\frac{r_2^2 - 3r_1^2}{r_2^2 - r_1^2} + \frac{4r_1^4}{(r_2^2 - r_1^2)^2}\log\frac{r_2}{r_1} + 4\log\frac{D}{r_2}\right.$$

$$+ \frac{4P}{(r_2^2 - r_1^2)^2}\left(r_2^4\log\frac{D^2}{D^2 - r_2^2} + r_1^4\log\frac{D^2}{D^2 - r_1^2} - 2r_1^2r_2^2\log\frac{r_2D^2}{r_1(D^2 - r_2^2)}\right)$$

$$\left. + \cdots\right]$$

(116)

$$+ \tfrac{1}{2}\mu_2\left[\frac{r_4^2 - 3r_5^2}{r_4^2 - r_5^2} + \frac{4r_5^4}{(r_4^2 - r_5^2)^2}\log\frac{r_4}{r_5} + 4\log\frac{D}{r_4}\right.$$

$$+ \frac{4P}{(r_4^2 - r_5^2)^2}\left(r_4^4\log\frac{D^2}{D^2 - r_4^2} + r_5^4\log\frac{D^2}{D^2 - r_5^2} - 2r_4^2r_5^2\log\frac{r_4D^2}{r_5(D^2 - r_4^2)}\right)$$

$$\left. + \cdots\right].$$

Comparing this solution with Maxwell's[3], one finds that the two agree except for the terms containing P. These latter are absent from Maxwell's solution, as is to be expected: for they are terms arising from the refraction of the lines of induction at the interfaces between the media of permeability μ_2 and the media of permeability μ_3, and this refraction Maxwell overlooked when effecting his solution.

III. We have

$$\mu_2 = \mu_4 \neq \mu_3, \qquad r_1 = r_5 = 0, \qquad U_2 = -U_4, \qquad \text{and}$$

$$d_{12} = d_{45} = 0.$$

[3] See Section 1.

Substituting these values for μ, U, and d in (89), we obtain

$$
\begin{aligned}
2W' = {} & \pi^2 \mu_3 w_2^2 r_2^4 - 4\pi^2 \mu_3 w_2^2 r_2^4 U_2 \\
& + \pi w_2 \sum_{n=1}^{\infty} (-1)^n r_2^2 \epsilon^{2nU_2} (M_n - Y_n) \\
& + \pi w_2 \sum_{n=1}^{\infty} (-1)^n r_2^2 (N_n - X_n).
\end{aligned} \tag{117}
$$

To determine the unknown coefficients in (117), we substitute the given values of μ, U, and d in (69) to (72) inclusive. Thus,

$$\left(\frac{\mu_3}{\mu_2} - 1\right) M_n \epsilon^{2nU_2} + \left(\frac{\mu_3}{\mu_2} + 1\right) Y_n = k_1, \tag{118}$$

$$-2X_n = 0, \tag{119}$$

$$-2N_n = 0, \tag{120}$$

$$\left(\frac{\mu_3}{\mu_2} + 1\right) M_n + \left(\frac{\mu_3}{\mu_2} - 1\right) Y_n \epsilon^{2nU_2} = k_4, \tag{121}$$

where

$$k_1 = -k_4 = 4\pi\mu_3 w_2 r_2^2 \frac{(-1)^n}{n} (\epsilon^{2nU_2} - 1). \tag{122}$$

For convenience, we rewrite this set of equations as follows:

$$PM_n \epsilon^{2nU_2} - Y_n = K, \tag{123}$$

$$-M_n + PY_n \epsilon^{2nU_2} = -K, \tag{124}$$

where

$$P = \frac{\mu_2 - \mu_3}{\mu_2 + \mu_3}, \tag{125}$$

$$I = \pi w_2 r_2^2, \tag{126}$$

$$K = -2\mu_3 I(P+1) \frac{(-1)^n}{n} (\epsilon^{2nU_2} - 1). \tag{127}$$

Solving (123) and (124) for M_n and Y_n, we obtain

$$M_n = -Y_n = K(P\epsilon^{2nU_2} + 1)^{-1}, \tag{128}$$

or, by virtue of (107),

$$M_n = -Y_n = K + K \sum_{m=1}^{\infty} (-P\epsilon^{2nU_2})^m. \tag{129}$$

Substituting in (117) as indicated and simplifying through use of (126), we

$$
\begin{aligned}
2W' = {} & \mu_2 I^2 - 4\mu_3 I^2 U_2 + 2I \sum_{n=1}^{\infty} (-1)^n K \epsilon^{2nU_2} \\
& + 2I \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-1)^n K (-P\epsilon^{2nU_2})^m \epsilon^{2nU_2}.
\end{aligned} \tag{130}
$$

Substituting (127) in (130) and dividing each member of the resulting expression by I^2, we have

$$L = \mu_2 - 4\mu_3 U_2 + 4\mu_3 \sum_{n=1}^{\infty} \frac{\epsilon^{2nU_2} - \epsilon^{4nU_2}}{n} + 4\mu_3 \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} (-1)^m \frac{\epsilon^{2nU_2} - 2\epsilon^{4nU_2} + \epsilon^{6nU_2}}{n} P^{m+1} \epsilon^{2nmU_2}. \tag{131}$$

Recalling from (112) that for $u < 0$

$$\sum_{n=1}^{\infty} \frac{\epsilon^{2nu}}{n} = -[u + \log \sinh(-u)],$$

we obtain from (131) after carrying out the necessary trigonometric transformations

$$L = \mu_2 + 4\mu_3 \left[\log \frac{D}{r_2} + P \log \frac{D^2}{D^2 - r_2^2} + P^2 \log \frac{D^2(D^2 - 2r_2^2)}{(D^2 - r_2^2)^2} + \tfrac{1}{2} P^3 \log \frac{D^4(D^2 - 2r_2^2)^4}{(D^2 - r_2^2)^3([D^2 - r_2^2] - D^2 r_2^2)^2} + \cdots \right]. \tag{132}$$

IV. If the conditions are the same as in III except that $r_2 \neq r_4$, the solution can be obtained directly from (132) by consideration of the physics of the problem. The total inductance will be the sum of two expressions, one due to each conductor, identical except for the interchange of r_2 with r_4. Accordingly, from (132) we obtain by inspection

$$\begin{aligned} L = {} & \tfrac{1}{2}\mu_2 + 2\mu_3 \left[\log \frac{D}{r_2} + P \log \frac{D^2}{D^2 - r_2^2} + P^2 \log \frac{D^2(D^2 - r_2^2 - r_4^2)}{(D^2 - r_2^2)^2} + P^3 \log \frac{D^2(D^2 - r_2^2 - r_4^2)^2}{(D^2 - r_2^2)([D^2 - r_2^2]^2 - D^2 r_4^2)} + \cdots \right] \\ & + \tfrac{1}{2}\mu_3 + 2\mu_3 \left[\log \frac{D}{r_4} + P \log \frac{D^2}{D^2 - r_4^2} + P^2 \log \frac{D^2(D^2 - r_2^2 - r_4^2)}{(D^2 - r_4^2)^2} + P^3 \log \frac{D^2(D^2 - r_2^2 - r_4^2)^2}{(D^2 - r_4^2)([D^2 - r_4^2]^2 - D^2 r_2^2)} + \cdots \right]. \end{aligned} \tag{133}$$

If (133) is compared with MacDonald's solution, it will be seen that the coefficients of the third power of P differ; the exponent of the second factor in the numerator of MacDonald's term is 2 smaller than in ours. Obviously, since this quantity must be dimensionless, our term is the right one; no doubt the exponent in MacDonald's solution is either an oversight on his part or an error introduced in setting type.

MacDonald's paper contains a still more serious error. If in the general expression for the inductance as given on page 260, we substitute $u = u'$, this expression for L does not reduce to the one found above; namely, to (131). It is not difficult to determine that a figure 4 should be placed before the summation sign in MacDonald's expression.

V. We have

$$\mu_2 \neq \mu_3 \neq \mu_4, \qquad r_1 = r_5 = 0, \qquad U_2 \neq -U_4, \qquad \text{and}$$
$$d_{12} = d_{45} = 0.$$

This is the most general case of the problem for solid conductors. MacDonald does not give a solution in terms of the various parameters, although one could be deduced by using his general expression for L, the same corrected as we have just indicated. Rather than work out the solution formally, we shall derive it from (133) inductively. We analyze as follows.

The terms containing powers of P are correcting terms due to the permeability of the conductors being different from the permeability of the surrounding medium. Now one method of obtaining correcting terms due to such a cause (although not a practical one for our particular configurations) is by "reflection"; that is, by application of Kelvin's image principle, first used by him in solving problems in electrostatics. If one conductor has the same permeability as the surrounding medium, there will be but one reflection, hence but one correction term. When both conductors have the same permeability, different from that of the surrounding medium, we get re-reflection in the two conductors, and we have an infinite number of correction terms. Further, because of the symmetry of the problem, the correction terms will be homogeneous in r_2 and r_4. These two deductions are confirmed by (133). Now, if the two conductors have different permeabilities as well as different radii, it follows that we will get a double infinity of correction terms, the two sets being identical except for the interchange of r_2 and r_4 and of P_2 and P_4, where

$$P_2 = \frac{\mu_2 - \mu_3}{\mu_2 + \mu_3} \quad \text{and} \quad P_4 = \frac{\mu_4 - \mu_3}{\mu_4 + \mu_3}.$$

Accordingly, we have from (133) by inspection

$$L = \tfrac{1}{2}\mu_2 + 2\mu_3 \log \frac{D}{r_2} + 2\mu_3 P_2 \log \frac{D^2}{(D^2 - r_2^2)} + \cdots$$
$$+ \tfrac{1}{2}\mu_4 + 2\mu_3 \log \frac{D}{r_4} + 2\mu_3 P_4 \log \frac{D^2}{(D^2 - r_4^2)} + \cdots. \tag{134}$$

4. Numerical analysis of III

Lord Rayleigh[9] pointed out that Maxwell's solution for steel lines in air neglected the effect of the field distortion due to the permeable conductors, and

[9] See Section 1.

stated, further, that "it would seem, however, that the error is of small importance when the wires are distant." It is of some interest, especially in view of the recent use of steel lines in rural electrification projects, to determine the order of magnitude of this error and at approximately what spacing it does become negligible. Accordingly, we assume[10] $\mu_2 = 20$, and compare the values computed from the first two terms of (132) with the corresponding values obtained by using the first four terms of (132), the third and fourth terms being of course, crrection terms for field flux distortion. These computations are, summarized in the table below.

TABLE 1

D	$\log \frac{D}{r_2}$	$P \log \frac{D^2}{D^2 - r_2^2}$	$P^2 \log \frac{D^2(D^2 - 2r_2^2)}{D^2 - r_2^2}$	First two terms of (132)	First four terms of (132)	Difference	% error
$2r_2$	.6931	.2603	−.1457	22.7724	23.2308	.4584	2
$3r_2$	1.0986	.1066	−.0130	24.3944	24.7688	.3744	1.5
$4r_2$	1.3863	.0584	−.0037	25.5452	25.7640	.2188	.8
$5r_2$	1.6094	.0255	−.0014	26.4376	26.5796	.1420	.5
$10r_2$	2.3026	.0091	−.0000	29.2104	29.2468	.0364	.1
$20r_2$	2.9957	.0023		31.9828	31.9920	.0092	—
$50r_2$	3.9120	.0004		35.6480	35.6495	.0014	—
$100r_2$	4.6052	.0001		38.4208	38.4212	.0004	—

Inspection of this table reveals that if the conductors are in contact an appreciable error is introduced when the inductance is computed, as is usual, from

$$L = \mu_2 + 4 \log \frac{D}{r_2}. \tag{135}$$

On the other hand, at normal transmission line spacings the effect of field distortion is negligible, and (135) can be used with confidence.

[10] One recognizes, of course, that actually the permeability is not constant over the cross-section of a steel line. Thus, in practice, one might take values approximating the minimum and maximum values of permeability and carry out computations as indicated. This would give limits between which the actual inductance would lie.

We have chosen $\mu = 20$, as tests carried out on steel wire for the Indiana Steel and Wire Company have shown that the permeability of the steel wire used for rural electrification has an approximate range of from 8 to 30, depending on the current density and frequency.

NOTE ON MEAN ONE-VALENT FUNCTIONS

By D. C. Spencer

1. Suppose that

$$f(z) = a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots \tag{1.1}$$

is regular for $|z| < 1$, and let d denote the lower bound of $|f(z)/z|$ for $|z| < 1$. In a recent paper[1] I showed that

$$d \geq \tfrac{1}{4}|a_1| \tag{1.2}$$

for strongly mean one-valent functions f, and pointed out that (1.2) is probably true throughout the wider class of mean one-valent functions. The best inequality of the form

$$d \geq \frac{|a_1|}{A} \tag{1.3}$$

for mean one-valent functions remains an unsolved problem.

Let F_p, where p is a positive integer,[2] denote the family of mean p-valent functions of the form

$$f(z) = a_p z^p + a_{p+1} z^{p+1} + \cdots . \tag{1.4}$$

The inequality (1.3) for functions of F_1 implies that

$$d \geq \frac{|a_p|}{A^p} \tag{1.5}$$

throughout F_p, $p > 1$—in particular throughout the subfamily of p-valent functions of form (1.4). The problem of the determination of A in (1.5) for p-valent functions, $p > 1$, has been discussed by several writers in recent years. Gelfer[3] has shown that $A \leq \dfrac{27 \log 3}{4} = 7.4156 \cdots$.

Using the method of M I shall show by computation that

$$A < 7 \tag{1.6}$$

in (1.3) (and therefore in (1.5)). At the same time enough additional data will be supplied to enable anyone to convince himself that a much better upper bound for A exists. In fact the method given may be used to obtain inequalities better than (1.6).

[1] D. C. Spencer, "On mean one-valent function," *Annals of Math.*, 42 (1941), 614–633. We shall refer to this paper as M.

[2] In the general theory of mean p-valent functions, p may be fractional.

[3] S. Gelfer, "Zur Theorie der multivalenten Funktionen," *Rec. Math.* [Mat. Sbornik] N.S. 8(50) (1940), 239–250.

2. For completeness we begin with a brief discussion of the method used in M. Suppose that the function

$$g(z) = b_1 z + b_2 z^2 + \cdots \tag{2.1}$$

is regular for $|z| < 1$, and maps the unit circle on a (Riemann) domain G. Let G^* be the "star" of G with respect to the point $g(0)$.[4] That is to say, G^* is that schlicht (one-valent) subdomain of G any point of which can be reached by starting at $g(0)$ and travelling always outward along a properly chosen radius.[5] Let $n^*(g)$ be the characteristic function of G^*—that is $n^*(g) = 1$ inside G^* and vanishes elsewhere—and define

$$p^*(R) = \frac{1}{2\pi}\int_{-\pi}^{\pi} n^*(Re^{i\Phi})\, d\Phi \qquad (g = Re^{i\Phi}). \tag{2.2}$$

Then we have[6]

$$2\pi \log |b_1| \leq 2\pi \int_0^\infty \log R\, d[-p^*(R)] = J^* \tag{2.3}$$

say.

To interpret the integral J^* geometrically, we observe that G^* contains the origin, hence some neighborhood of the origin. Let R_0 be so small that the circle with center at $g = 0$ and radius R_0 lies entirely in the interior of G^*. Then $p^*(R) = 1$, and so $d[-p^*(R)] = 0$, for $0 \leq R \leq R_0$. It follows that

$$\begin{aligned} J^* &= 2\pi \int_0^\infty \log R\, d[-p^*(R)] = 2\pi \int_{R_0}^\infty \log R\, d[-p^*(R)] \\ &= [-2\pi \log R \cdot p^*(R)]_{R_0}^\infty + \int_{R_0}^\infty p^*(R)\, d(2\pi \log R) \\ &= \int_{R_0}^\infty p^*(R)\, d(2\pi \log R) - 2\pi \log \frac{1}{R_0} \end{aligned} \tag{2.4}$$

since $p^*(R_0) = 1$, $p^*(\infty) = 0$.

We remove the circle of radius R_0 and center $g = 0$ from G^*, make the resulting domain simply-connected by means of a slit extending along the real axis from R_0 to the boundary, and map it by the function $\log g$ into a strip, S^*. Then we see that

$$\int_{R_0}^\infty p^*(R)\, d(2\pi \log R) = \text{area of } S^* = A(S^*) \tag{2.5}$$

say. If in particular G^* is the unit circle, we denote the corresponding S^* by T^*, so that $A(T^*) = 2\pi \log \frac{1}{R_0}$. By (2.4):

$$J^* = A(S^*) - A(T^*); \tag{2.6}$$

[4] The projection of $g(0)$ on the complex plane is of course 0.

[5] In intuitive language G^* is the subdomain of G visible from $g(0)$.

[6] See M.

that is to say J^* is the difference between the logarithmic area of G^* and that of the unit circle.

3. Let W be the map of $|z| < 1$ by $f(z)$. f is said to be mean one-valent if the area (multiply-covered regions being counted multiply) of that portion of W which lies inside the circle $|w| < R$ does not exceed πR^2 for all $R > 0$. Mean one-valency implies, in particular, that $d > 0$ and that the circle $|w| < d$ is simply covered.[7]

We write

$$K(z) = \frac{4z}{(1+z)^2} \tag{3.1}$$

and denote by $K^{-1}(w)$ that branch of the function inverse to K which vanishes at the origin. K maps $|z| < 1$ onto the w-plane slit from 1 to ∞ along the real axis.

By making a trivial Euclidean transformation, we may suppose that the distance d of the boundary of W from $w = 0$ is 1, and that $f \neq 1$ in $|z| < 1$.[8] Then the function

$$g(z) = K^{-1}\{f(z)\} = b_1 z + b_2 z^2 + \cdots, \qquad |b_1| = \tfrac{1}{4}|a_1|, \tag{3.2}$$

is regular for $|z| < 1$, and we may apply (2.3). The problem is to find an upper bound for J^* from which we can infer that $|a_1| < 7$—a result equivalent to $d > \frac{1}{7}|a_1|$ since by supposition $d = 1$.

4. We define

$$W^+ = K(G^*). \tag{4.1}$$

Denoting by $t = t(\varphi)$ the transform by $K(z)$ of the ray $\arg z = \varphi$, $0 \leq |z| < \infty$, we see that W^+ is the subdomain of W any point P of which can be reached by starting from $w = 0$ and then travelling outward along some curve t. Curves t which intersect the circumference $|w| = 1$ at intervals of 5° from 0° to 180° are shown in Figs. 1a and 1b (Appendix). Only the upper half-plane is represented since the function is symmetrical with respect to the real axis, and furthermore, the portions of these lines inside the unit circle are omitted.

Suppose first that $\varphi \neq 0$ or π. Then as z moves outward along the ray $\arg z = \varphi$ from $z = 0$ to $z = e^{i\varphi}$, the image point $w = K(z)$ moves from $w = 0$ along t until it meets the positive real axis. As z moves along $\arg z = \varphi$ from $z = e^{i\varphi}$ to ∞, w describes the conjugate path in the lower half-plane back to $w = 0$. If $\varphi = 0$, w moves along the positive real axis to 1, then back again to the origin. Finally if $\varphi = \pi$, w moves along the negative real axis from 0 to ∞, then back to 0. In other words, K maps the simple z-plane onto the two-sheeted w-plane with branch points at 1 and ∞.

[7] See *M*.
[8] See *M*.

Let $n^+(w)$ be the number of times W^+ covers the point w. Since $d = 1$, we see that $n^+(w)$ assumes only the values 0, 1, 2, and that $n^+(w) = 1$ for $0 \leq |w| < 1$.

Let da_w, da_g be corresponding elements of area in the w and $\log g$ planes, and let R, $\Re$ be the distances from da_w to $w = 0$ and $w = 1$ respectively. Then (see M):

$$da_g = \frac{da_w}{R^2 \Re}. \tag{4.2}$$

Writing $w = Re^{i\Phi}$, it is easy to see from (4.2) that, subject to mean one-valency,

$$J^* \leq J_0 \tag{4.3}$$

where J_0 is the value of the integral J^* for

$$n^+(Re^{i\Phi}) = \begin{cases} 1, & 0 \leq R \leq 1 \\ 2, & R > 1, \quad -\frac{\pi}{2} < \Phi < \frac{\pi}{2} \\ 0, & R > 1, \quad \frac{\pi}{2} \leq \Phi \leq \frac{3\pi}{2}. \end{cases} \tag{4.4}$$

We now obtain an upper bound for J_0.

5. Let u be the half of a curve t which lies in the upper half-plane, and let u_α denote that curve u which intersects the unit circumference $|w| = 1$ at the point $w = e^{i\frac{\alpha}{180}\pi}$, α being measured in degrees. The curves $u_5, u_{10}, \cdots, u_{180}$ are shown in Figs. 1a and 1b. We denote by $\Delta_{\alpha,\beta}$ $(\beta > \alpha)$ the domain of points w, $|w| > 1$, in the upper half-plane bounded by u_α and u_β, and denote by $a_w(R_1, R_2 \mid \alpha, \beta)$ the area of the portion of $\Delta_{\alpha,\beta}$ contained in the annulus $R_1 < |w| < R_2$. Finally let $a_g(R_1, R_2 \mid \alpha, \beta)$ be the area into which a_w is transformed by $\log g = \log K^{-1}(f)$.

The areas a_w, a_g defined by the curves u_α, $\alpha = 5m$ $(m = 1, 2, \cdots, 36)$, and $|w| = R$ $(R = 1, 2, \cdots, 10, 15, 20, 25, 30, 40, 50, 60)$ are tabulated in Table I of the Appendix. Areas a_w are shown in Figs. 1a and 1b: areas a_g in Fig. 2. In the tabulation $a_w(R_1, R_2 \mid \alpha, \alpha + 5)$ is the upper, $a_g(R_1, R_2 \mid \alpha, \alpha + 5)$ the lower, number in row (R_1, R_2), column $(\alpha, \alpha + 5)$. For example $a_w(3, 4 \mid 155, 160) = .469$, $a_g(3, 4 \mid 155, 160) = .009$.

6. Now it is plain that J^* vanishes for $n^+(w)$ identically equal to unity. For in this case we may suppose that $f = K$, and so $g = z$. Hence if J_z^* is the value of J^* for $f = z$, $J_z^* < 0$. In fact

$$J_z^* = -2a_g(1, \infty \mid 0, 180) < -7.252 \tag{6.1}$$

as can be seen, for example, by summing all the areas a_g in Table I.[9]

[9] To avoid the possibility of an accumulated error, the estimation (6.1) was actually obtained in another way.

Let $\mathfrak{I}^*$ be the contribution to J^* arising from the portion of W^+ lying in the region $|w| > 1$. Then

$$J^* = \mathfrak{I}^* + J_*^* < \mathfrak{I}^* - 7.252 \tag{6.2}$$

by (6.1). Similarly if J_0 is the contribution to J_0 of the portion of the distribution (4.4) for which $|w| = R > 1$, we plainly have

$$\mathfrak{I}^* < \mathfrak{I}_0 . \tag{6.3}$$

An inspection of Figs. 1a and 1b now shows that $\mathfrak{I}_0$ is *less* than four times the sum of the following areas a_β: (1, 2 | 0, 110), (2, 3 | 60, 120), (3, 4 | 80, 130), (4, 5 | 95, 135), (5, 6 | 105, 135), (6, 7 | 110, 140), (7, 8 | 115, 145), (8, 9 | 120, 145), (9, 10 | 125, 145), (10, 15 | 125, 155), (15, 20 | 135, 155), (20, 25 | 140, 160), (25, 30 | 145, 160), (30, 40 | 150, 165), (40, 50 | 150, 165), (50, 60 | 155, 170), (60, ∞ | 155, 180). This sum is 10.784, and a slightly less wasteful estimate shows that $\mathfrak{I}_0 < 10.784$ even when the error introduced in summation is allowed for. Hence by (6.2) and (6.3)

$$J^* < 3.532 \tag{6.4}$$

from which it follows that

$$|a_1| < 7. \tag{6.5}$$

7. A little consideration shows that no W^+ satisfying (4.4) exists, and that

$$J^* < J_0 . \tag{7.1}$$

If only topologically possible distributions W^+ are admitted, the data provided in Table I indicates that the correct upper bound for the constant A is considerably smaller than 7.

8. I take this opportunity to correct the following errors in M: Read

p. 629, line 9 from the foot: log $K_\beta^{-1}(w)$ for $K_\beta^{-1}(w)$.

p. 629, line 3 from the foot: 'increasing' for 'decreasing.'

p. 630, line 3: 'J_1 is not less than the area J_0' for 'J_1 is the area.'

p. 630, line 4 from the foot: 'arg' for 'any.'

p. 630, line 3 from the foot: J_0 for J_1.

p. 630, line 1 from the foot: J^* for J, and J_0 for J_1.

p. 631, line 13 from the foot: J_0 for J_1. Delete the sentence beginning 'Finally . . . .'

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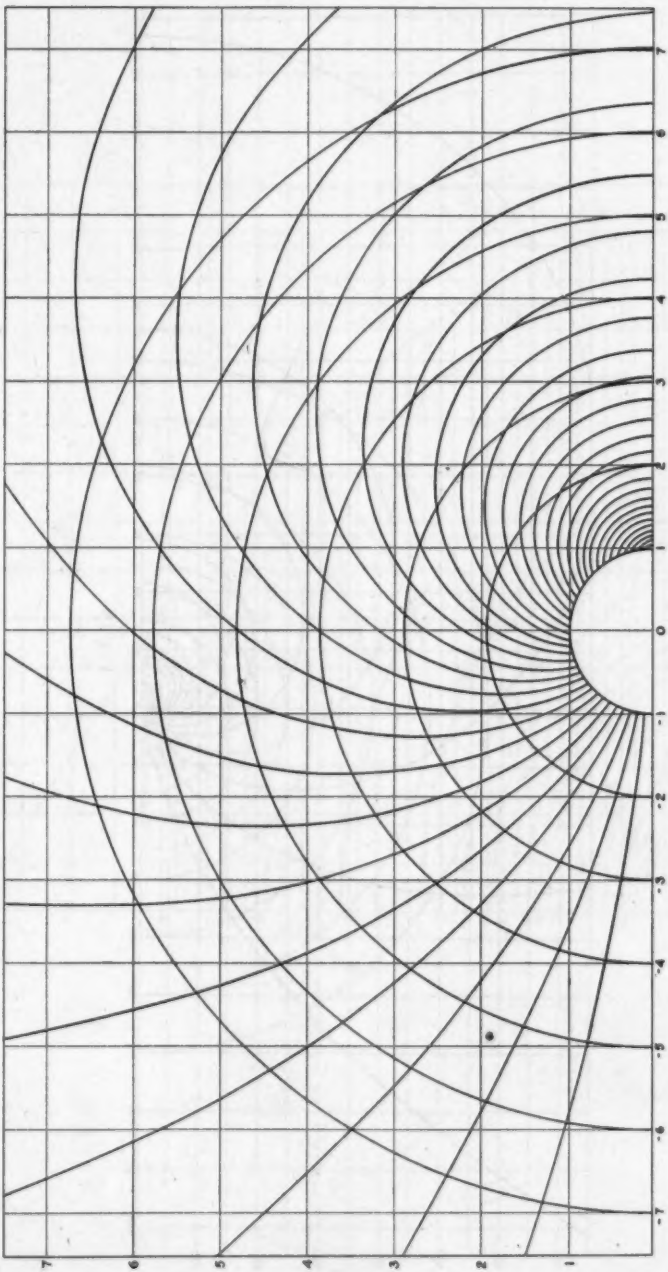

FIG. 1a

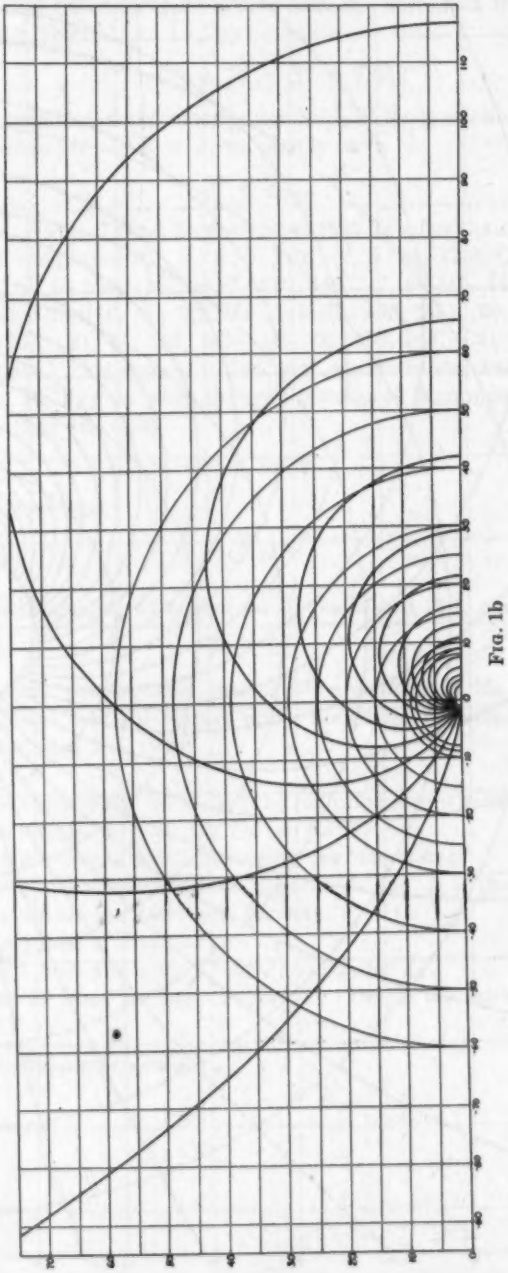

FIG. 1b

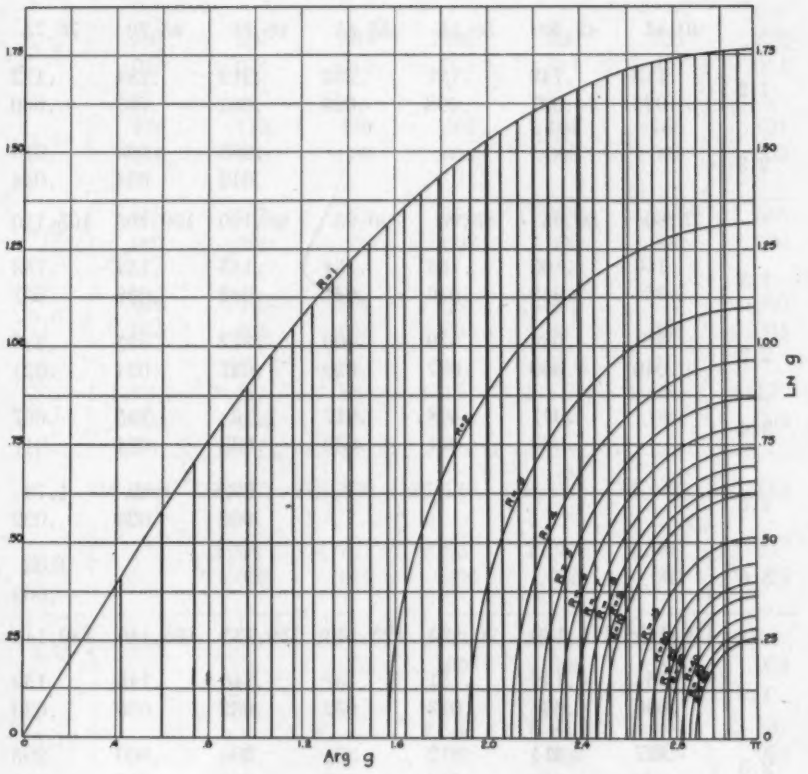
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FIG. 2

Appendix

TABLE I

	0,5	5,10	10,15	15,20	20,25	25,30	30,35	35,40
1,2	.003	.009	.017	.025	.035	.050	.066	.085
	.088	.089	.090	.091	.093	.093	.095	.096

	40,45	45,50	50,55	55,60	60,65	65,70	70,75
1,2	.110	.142	.181	.232	.212	.184	.172
	.096	.097	.098	.099	.081	.067	.060
2,3					.086	.198	.318
					.019	.034	.044

	75,80	80,85	85,90	90,95	95,100	100,105	105,110
1,2	.166	.160	.164	.154	.155	.153	.153
	.054	.048	.046	.041	.042	.038	.037
2,3	.471	.634	.479	.409	.372	.355	.335
	.048	.050	.037	.029	.027	.024	.021
3,4		.041	.468	.917	1.16	.795	.667
		.003	.021	.030	.033	.021	.017
4,5					.323	1.49	1.73
					.006	.023	.022
5,6							1.02
							.009

	110,115	115,120	120,125	125,130	130,135	135,140	140,145
1,2	.154	.149	.151	.151	.146	.148	.150
	.036	.034	.033	.032	.032	.030	.030
2,3	.327	.314	.312	.304	.304	.300	.295
	.020	.019	.018	.017	.016	.015	.015
3,4	.603	.567	.537	.521	.503	.494	.481
	.015	.013	.012	.012	.011	.010	.010
4,5	1.12	.960	.851	.800	.747	.740	.699
	.014	.011	.010	.009	.008	.007	.007
5,6	2.72	1.71	1.34	1.17	1.08	1.01	.922
	.019	.011	.009	.007	.006	.006	.006
6,7	.711	3.65	2.22	1.72	1.48	1.36	1.27
	.004	.015	.009	.007	.006	.005	.005

TABLE I (*Continued*)

	145,150	150,155	155,160	160,165	165,170	170,175	175,180
1,2	.147	.149	.148	.148	.148	.151	.146
	.029	.029	.029	.028	.027	.028	.028
2,3	.297	.297	.286	.292	.290	.290	.292
	.015	.014	.014	.014	.014	.014	.014
3,4	.479	.476	.469	.462	.464	.460	.461
	.010	.009	.009	.009	.008	.009	.009
4,5	.688	.683	.672	.656	.663	.654	.655
	.007	.006	.006	.006	.006	.006	.006
5,6	.938	.919	.897	.884	.871	.872	.866
	.005	.005	.005	.005	.005	.004	.004
6,7	1.22	1.18	1.16	1.13	1.11	1.09	1.12
	.004	.004	.004	.004	.004	.004	.004

	110,115	115,120	120,125	125,130	130,135	135,140	140,145
7,8		1.03	4.67	2.62	2.06	1.79	1.64
		.003	.013	.006	.005	.004	.004
8,9			2.60	4.93	2.86	2.29	2.14
			.005	.008	.005	.004	.004
9,10				6.02	4.02	2.96	2.53
				.008	.005	.004	.003
10,15				2.25	20.3	38.3	24.3
				.002	.015	.019	.013
15,20						9.91	64.5
						.003	.013
20,25							12.0
							.002

TABLE I (*Concluded*)

	145,150	150,155	155,160	160,165	165,170	170,175	175,180
7,8	1.52	1.45	1.42	1.32	1.38	1.35	1.35
	.004	.003	.003	.003	.003	.003	.003
8,9	1.88	1.79	1.67	1.58	1.67	1.65	1.65
	.003	.003	.003	.003	.003	.003	.003
9,10	2.29	2.14	2.04	1.96	2.02	1.93	1.93
	.003	.003	.002	.002	.002	.002	.002
10,15	19.4	17.5	16.1	15.2	14.8	14.0	14.3
	.010	.009	.008	.008	.007	.007	.007
15,20	42.7	32.0	27.9	25.7	24.9	23.7	23.6
	.008	.006	.005	.005	.004	.004	.004
20,25	98.7	55.9	44.0	38.8	35.6	34.5	33.9
	.009	.005	.004	.003	.003	.003	.003
	145,150	150,155	155,160	160,165	165,170	170,175	175,180
25,30	57.8	113.	65.1	54.4	48.9	46.6	45.8
	.003	.005	.003	.003	.002	.002	.002
30,40		285.	235.	169.	145.	134.	131.
		.007	.006	.004	.003	.003	.003
40,50		27.1	505.	275.	219.	198.	190.
		.000	.006	.003	.002	.002	.002
50,60			451.	440.	310.	270.	257.
			.003	.003	.002	.002	.002
60,∞			∞	∞	∞	∞	∞
			.001	.009	.013	.015	.016

THE LATTICE POINTS OF TETRAHEDRA

By D. C. Spencer

1. This note is a sequel to my paper "On a Hardy-Littlewood problem of Diophantine approximation," *Proc. Cambridge Phil. Soc.*, XXXV (1939), 527–547, in which the two dimensional problem was considered. The consideration of higher dimensions leads to interesting new problems in an unexplored domain of Diophantine analysis. My purpose here is to point out a few of these problems.

2. Suppose that $\eta, \omega_1, \omega_2, \cdots, \omega_s$ are positive real numbers and that $r \geq 0$. Let

$$N_r^{(s)}(\eta) = \Sigma(\eta - m_1\omega_1 - \cdots - m_s\omega_s), \tag{2.1}$$

where the summation is over all $m_k \geq 1$ $(k = 1, 2, \cdots, s)$ for which

$$m_1\omega_1 + \cdots + m_s\omega_s \leq \eta. \tag{2.2}$$

Then $N_0^{(s)}(\eta)$ is the number of solutions of the inequality (2.2). In geometrical language $N_0^{(s)}(\eta)$ is the number of lattice points lying inside or on the boundary of the s-dimensional tetrahedron bounded by the s coördinate hyperplanes $x_1 = 1, x_2 = 1, \cdots, x_s = 1$ and the hyperplane

$$\omega_1 x_1 + \omega_2 x_2 + \cdots + \omega_s x_s = \eta. \tag{2.3}$$

$N_r^{(s)}(\eta)$ is the r − pl (perhaps fractional) integral of $N_0^{(s)}(\eta)$.

Suppose that $\alpha \geq 0$ and let $\lambda_1, \lambda_2, \cdots, \lambda_n, \cdots$ $(\lambda_n < \lambda_{n+1})$ be the set of numbers which are representable in the form

$$\lambda_n = \alpha + m_1\omega_1 + \cdots + m_s\omega_s \tag{2.4}$$

where $m_k \geq 0$ $(k = 1, 2, \cdots, s)$. Let a_n be the number of representations of λ_n. Then for $Re(z) > 0$

$$\frac{e^{-\alpha z}}{\prod_{k=1}^{s}(1 - e^{-\omega_k z})} = \sum_1^\infty a_n e^{-\lambda_n z} \tag{2.5}$$

where the Dirichlet's series is absolutely convergent. Taking $\alpha = \sum_1^s w_k$ we have by Perron's formula[1]

$$N_r^{(s)}(\eta) = \frac{\Gamma(1+r)}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{(\eta - \sum_1^s \omega)z}}{z^{1+r}\prod_1^s (1 - e^{-\omega_k z})}\, dz \tag{2.6}$$

[1] See G. H. Hardy and M. Riesz, "The general theory of Dirichlet's series," Cambridge Tract No. 18.

where $c > 0$. Here we suppose that $z^{1+r} = \exp\{(1+r)\log z\}$, that branch of $\log z$ being taken which is real on the real axis. When $r = 0$ the integral is defined as

$$\lim_{T\to\infty} \int_{c-iT}^{c+iT} , \tag{2.7}$$

in which case lattice points on the hyperplane (2.3) are counted with weight $\frac{1}{2}$.

Deforming the line of integration in (2.6) into a contour embracing the real axis and containing no singularity of the integrand other than the origin, we see that

$$N_r^{(s)}(\eta) = (-1)^s \zeta_s(-r, \eta \mid \omega_1, \omega_2, \cdots, \omega_s) + T_r^{(s)}(\eta) \tag{2.8}$$

where ζ_s is the s-pl ζ-function of Barnes[2] and $T_r^{(s)}(\eta)$ is a sum of terms resulting from the residues at the poles of the integrand which lie on the imaginary axis. When r is integral, $\zeta_s(-r, \eta)$ reduces to a polynomial of degree $s + r$ whose coefficients are symmetric functions of the ω's; in general $\zeta_s(-r, \eta)$ has the form

$$K_0\eta^{s+r} + K_1\eta^{s+r-1} + \cdots + K_{s+[r]}\eta^{(r)} + O(1)$$

where $(r) = r - [r]$.

The problem of approximating to $N_0^{(s)}(\eta)$ by polynomials has been considered by D. H. Lehmer[3]—in particular the polynomial

$$R_s(\eta \mid \omega_1, \omega_2, \cdots, \omega_s) = (-1)^s \zeta_s(0, \eta \mid \omega_1, \omega_2, \cdots, \omega_s) \tag{2.9}$$

is compared numerically in special cases with several others constructed by Lehmer, J. B. Rosser, and A. E. Western. We shall see below (Theorem II) that for almost all $(\omega_1, \omega_2, \cdots, \omega_s)$ R_s is asymptotically the best approximating polynomial.

We note in passing the two relations:

$$N_r^{(s)}(\eta + \omega_s) - N_r^{(s)}(\eta) = N_r^{(s-1)}(\eta), \tag{2.10}$$

$$N_r^{(s)}(\eta) = \sum_{m\geq 1} N_r^{(s-1)}(\eta - m\omega_s), \tag{2.11}$$

immediate consequences of (2.6) but obvious *a priori* from geometrical considerations.

3. The delicate features of the problem appear in connection with the term $T_r^{(s)}(\eta)$, the behavior of which is entirely dependent on the arithmetical nature of the ω's.

We divide the ω's into n classes C_k $(k = 1, 2, \cdots, n)$, ω and ω' belonging to the same class if and only if the ratio ω/ω' is rational, and denote the ω's of C_k by $\omega_1^{(k)}, \omega_2^{(k)}, \cdots, \omega_{m_k}^{(k)}$. The numbers $\omega_1^{(k)}, \omega_2^{(k)}, \cdots, \omega_{m_k}^{(k)}$ are either all rational

[2] E. W. Barnes, "On the theory of the multiple Gamma function," *Trans. Cambridge Phil. Soc.*, XIX (1904), 374–425.

[3] D. H. Lehmer, "The lattice points of an n-dimensional tetrahedron," *Duke Math. Journal*, 7(1940), 341–353.

or all irrational; if all irrational each is a rational multiple of $\omega_1^{(k)}$. Therefore in either case these numbers are of essentially the same arithmetical structure.

If $n > 1$ let $f_j(x)$ $(j = 1, 2, \cdots, n)$ be a continuous increasing function satisfying

$$\prod_{\substack{k=1 \\ k \neq j}}^{n} \left| \sin \frac{\lambda \omega_1^{(k)} \pi}{\omega_1^{(j)}} \right|^{m_k} \geq \frac{1}{f_j(\lambda)} \tag{3.1}$$

for all integral $\lambda \geq 1$, and let $g_j(x)$ be the function inverse to f_j. If $n = 1$ we define $g(x) = g_1(x) = 1$. It is probable that the following inequality is generally valid:

$$T_r^{(s)}(\eta) = O\left[\sum_{j=1}^{n} \eta^{m_j - 1} \cdot \max\left\{\frac{\eta^{s-m_j}}{[g_j(\eta^{s-m_j})]^{1+r}}, [\nu_r(\eta)]^{s-m_j}\right\}\right] \tag{3.2}$$

where

$$\nu_0(\eta) = \log \eta, \tag{3.3}$$

and

$$\nu_r(\eta) = \begin{cases} 1, & \text{if, for some } \epsilon > 0, \quad \dfrac{\eta^{s-m_j}}{[g_j(\eta^{s-m_j})]^{1+r}} = O(\eta^{-\epsilon}) \\ \log \log \eta, & \text{otherwise} \end{cases} \tag{3.4}$$

if $r > 0$.

4. If for all $\epsilon > 0$

$$g_j(x) < x^{\epsilon}, \tag{4.1}$$

that is to say if $g_j(x)/x^{\epsilon} \to 0$ as $x \to \infty$, the inequality (3.2) is true. But if (4.1) is satisfied, the analysis is neither interesting nor difficult, and we therefore omit a proof of (3.2) in this case.

We have therefore, in particular, the following (best possible) result:

Theorem I. *If* $n = 1$,

$$T_r^{(s)}(\eta) = O(\eta^{s-1}); \tag{4.2}$$

if $n > 1$,

$$T_r^{(s)}(\eta) = 0(\eta^{s-1}). \tag{4.3}$$

It is perhaps worthwhile to point out another result of this type for the case in which $\omega_k = \log p_k$, where $p_1, p_2, \cdots, p_s$ are distinct primes. Then $n = s$, and for any $\epsilon > 0$ we may take[4]

$$f_j(x) = Ae^{\epsilon x} \qquad (j = 1, 2, \cdots, s)$$

[4] See G. H. Hardy, *Ramanujan*, Cambridge University Press, p. 78.

where A depends on ϵ but not on x. Since (4.1) is satisfied, we have by (3.2):

$$T_r^{(s)}(\eta) = 0\left\{\frac{\eta^{s-1}}{(\log \eta)^{1+r}}\right\}. \tag{4.4}$$

This inequality, though certainly far from the real truth, is interesting for the reason that it is the best that can be deduced from the known arithmetical properties of the logarithms of primes.

5. The main interest attaches to the cases where $T_r^{(s)}(\eta)$ is relatively small, and in this direction the following result is not difficult to prove:

Theorem II. *For almost all points* $\Omega = (\omega_1, \omega_2, \cdots, \omega_s)$ *and any* $\epsilon > 0$:

$$T_0^{(s)}(\eta) = O\{(\log \eta)^{s+\epsilon}\}; \tag{5.1}$$

$$T_r^{(s)}(\eta) = O(1) \quad (r > 0). \tag{5.2}$$

Since the Ω for which $n < s$ are obviously of measure zero, we may suppose in Theorem II that $n = s$. Now it is not difficult to prove that for any $\epsilon > 0$ and almost all points $\Theta = (\theta_1, \theta_2, \cdots, \theta_k)$ in k-dimensional space

$$\prod_{j=1}^{k} \left|\sin m\pi\theta_j\right| \geq \frac{K}{m \log^{k+\epsilon} m} \qquad (m \geq 1;\ k \geq 1 \text{ and fixed}), \tag{5.3}$$

where K depends on Θ and k but not on m. Therefore if we assume (3.2) we have in place of (5.1) the stronger inequality

$$T_0^{(s)}(\eta) = O\{(\log \eta)^{s-1+\epsilon}\}. \tag{5.4}$$

But (5.4) has not been proved for $s > 2$.

6. The term $T_r^{(s)}(\eta)$ is the sum of the residues at the poles, other than the origin, of the function

$$\Gamma(1+r)\frac{e^{(\eta-\Sigma\omega)z}}{z^{1+r}\prod_1^s (1-e^{-\omega_k z})}. \tag{6.1}$$

The multiplicity p of the pole at $z = 2\lambda\pi i/\omega_{\nu_1}^{(j)}$ (λ integral) is equal to the number of the $\omega_\mu^{(j)}$ for which $\lambda\omega_\mu^{(j)}/\omega_{\nu_1}^{(j)}$ is an integer, and therefore $1 \leq p \leq m_j$. We denote this set of $\omega_\mu^{(j)}$ by $(\omega_{\nu_1}^{(j)}, \omega_{\nu_2}^{(j)}, \cdots, \omega_{\nu_p}^{(j)})$. Combining the residue at this pole with that at the conjugate pole $z = -2\lambda\pi i/\omega_{\nu_1}^{(j)}$, we find that the dominant term of the residue is

$$\left\{\frac{\Gamma(1+r)}{2^{s+r-p}\pi^{1+r}\Gamma(p)} \cdot \frac{(\omega_{\nu_1}^{(j)})^{1+r}}{\prod_{l=1}^{p}\omega_{\nu_l}^{(j)}} \cdot \frac{\cos\left(\frac{2\lambda\pi}{\omega_{\nu_1}^{(j)}}\left(\eta - \frac{1}{2}\sum_1^s \omega\right) - (s-p+r+1)\pi\right)}{\lambda^{1+r}\prod_{k=1}^{n}\prod_{\substack{\mu_k=1 \\ \mu_k \neq \nu_l}}^{m_k} \sin\frac{\lambda\omega_{\mu_k}^{(k)}\pi}{\omega_{\nu_1}^{(j)}}}\right\}\eta^{p-1}.$$

The other terms give contributions of orders $\eta^{p-2}, \eta^{p-3}, \cdots, \eta$, and $O(1)$. Summing over all p-pl poles it is easily shown that the resulting contribution to $T_r^{(s)}(\eta)$ is at least of order η^{p-1} for a sequence of η tending to infinity.

If $n = s$, all poles (other than the origin) are simple, and in this case writing

$$(6.2) \qquad (\varphi(\omega_1, \omega_2, \cdots, \omega_s))^* = \sum_{j=1}^{s} \varphi(\omega_j, \omega_{j+1}, \cdots \omega_s, \omega_1, \omega_2, \cdots, \omega_{j-1})$$

we have:

Theorem III. *If $n = s$,*

$$(6.3) \quad T_r^{(s)}(\eta) = \frac{\Gamma(1+r)}{2^{r+s-1}\pi^{1+r}} \left(\omega_j^r \sum_{\lambda_j=1}^{\infty} \frac{\cos\left(\frac{2\lambda_j\pi}{\omega_j}\left(\eta - \frac{1}{2}\sum_1^s \omega\right) - \frac{1}{2}(r+s)\pi\right)}{\lambda_j^{1+r} \prod_{\substack{k=1 \\ k\neq j}}^{s} \sin \frac{\lambda_j \omega_k \pi}{\omega_j}} \right)^*.$$

The summation is to be effected as follows: the partial sums are formed of those terms of the s series for which $\lambda_j < \frac{\omega_j Y}{2\pi}$ $(j = 1, 2, \cdots, s)$, and the limit is taken as $Y \to \infty$ through a sequence (Y_j) all of whose members differ by a number $K(\omega_1, \omega_2, \cdots, \omega_s)$ from any of the numbers $\frac{2\lambda_j\pi}{\omega_j}$.

7. We digress in order to make one or two remarks of a formal nature, and we suppose always that $n = s$. First, if $0 < \eta < \sum_1^s \omega$, $N_r^{(s)}(\eta) = 0$, and so by (2.8) for this range of η

$$\zeta_s(-r, \eta \mid \omega_1, \omega_2, \cdots, \omega_s)$$

$$(7.1) \qquad = (-1)^{r-1} \frac{\Gamma(1+r)}{2^{r+s-1}\pi^{1+r}} \left(\omega_j^r \sum_{\lambda_j=1}^{\infty} \frac{\cos\left(\frac{2\lambda_j\pi}{\omega_j}(\eta - \tfrac{1}{2}\textstyle\sum\omega) - \tfrac{1}{2}(r+s)\pi\right)}{\lambda_j^{1+r} \prod_{k\neq j} \sin \frac{\lambda_j \omega_k \pi}{\omega_j}} \right)^*$$

$$= (S^{(j)})^*$$

say. Secondly, writing $\eta = \eta_1 + \eta_2 + \cdots + \eta_s$, we suppose that $0 < \eta_k < \omega_k$. It can be shown that ζ_s is expressible as an s-pl Fourier series:

$$\zeta_s(-r, \eta \mid \omega_1, \omega_2, \cdots, \omega_s)$$

$$(7.2) \qquad = \sum_{-\infty}^{+\infty} c_{\mu_1, \mu_2, \cdots \mu_s} \exp\left\{-2\pi i \left(\frac{\mu_1\eta_1}{\omega_1} + \frac{\mu_2\eta_2}{\omega_2} + \cdots + \frac{\mu_s\eta_s}{\omega_s}\right)\right\}$$

where

$$c_{\mu_1,\mu_2,\cdots,\mu_s} = c_{\mu_1,\mu_2,\cdots,\mu_s}(-r \mid \omega_1, \omega_2, \cdots, \omega_s)$$

$$(7.3) \qquad = \frac{\Gamma(1+r)}{(2\pi)^{r+s}(i)^{s-1}} \left(\omega_j^r \frac{\exp\left\{\frac{(1+r)\pi s}{2}\right\}}{\mu_j^{1+r} \prod_{\substack{1 \\ k \neq j}}^{s} \left(\mu_j \frac{\omega_k}{\omega_j} - \mu_k\right)} \right)^* = (c^{(j)}_{\mu_1,\mu_2,\cdots,\mu_s})^*,$$

say. (7.1) and (7.2) are connected by the formal identity

$$(7.4) \qquad \sum_{-\infty}^{+\infty} c^{(j)}_{\mu_1,\mu_2,\cdots,\mu_s} \exp\left\{-2\pi i \left(\frac{\mu_1\eta_1}{\omega_1} + \frac{\mu_2\eta_2}{\omega_2} + \cdots + \frac{\mu_s\eta_s}{\omega_s}\right)\right\} = S^{(j)},$$

both sides of which may be infinite. These relations were pointed out by Hardy[6] in the case $s = 2$.

Simple transformations of (6.3) yield different forms. For example,

$$(7.5) \qquad T_0^{(2)}(\eta) = -\frac{\omega_1\omega_2}{\pi^2} \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \left(\frac{\cos 2\pi m_1\eta/\omega_1 - \cos 2\pi m_2\eta/\omega_2}{m_1^2\omega_2^2 - m_2^2\omega_1^2} \right) + O(1)$$

if the summation is effected in the right way. One is also led to (7.5) by a direct summation of lattice points using Euler's summation formula.

8. We now prove Theorem II, and base the proof on the following lemma:

Lemma I. *The series*

$$(8.1) \qquad \sum_{m=2}^{\infty} \frac{1}{m \log^{k+1+\epsilon} m \prod_1^k |\sin m\pi\theta_j|}$$

is convergent for every $\epsilon > 0$ *and almost all* $(\theta_1, \theta_2, \cdots, \theta_k)$.

In fact

$$\int_0^1 \cdots \int_0^1 \sum_2^{\infty} \frac{d\theta_1 d\theta_2 \cdots d\theta_k}{m \log^{1+\epsilon} m \prod_1^k |\sin m\pi\theta_j| \cdot |\log|\sin m\pi\theta_j||^{1+\epsilon}}$$

$$< K(\epsilon) \sum_2^{\infty} \frac{1}{m \log^{1+\epsilon} m} < K(\epsilon)$$

since

$$\int_0^1 \frac{d\theta_j}{|\sin m\pi\theta_j| \cdot |\log|\sin m\pi\theta_j||^{1+\epsilon}} < K(\epsilon).$$

Hence for almost all $\Theta = (\theta_1, \theta_2, \cdots, \theta_k)$ the series

$$\sum_2^{\infty} \frac{1}{m \log^{1+\epsilon} m \prod_1^k |\sin m\pi\theta_j| \cdot |\log|\sin m\pi\theta_j||^{1+\epsilon}}$$

[6] G. H. Hardy, "On double Fourier series, and especially those which represent the double Zeta-function with real and incommensurable parameters," *Quart. Journal*, XXXVII (1905), 53–79.

is convergent. From this it follows in particular that for almost all Θ there exists a $K(\Theta)$ such that

$$|\sin m\pi\theta_j| \geq Km^{-2} \qquad (j = 1, 2, \cdots, k),$$

and so

$$\left|\prod_1^k \log|\sin m\pi\theta_j|\right|^{1+\epsilon} \leq K \log^{k(1+\epsilon)} m.$$

Therefore for $\epsilon' = \epsilon/(k+1)$ and almost all Θ we have

$$\sum_2^\infty \frac{1}{m \log^{k+1+\epsilon} m \prod_1^k |\sin m\pi\theta_j|}$$

$$\leq K \sum_2^\infty \frac{1}{m \log^{1+\epsilon'} m \prod_1^k |\sin m\pi\theta_j| \cdot |\log|\sin m\pi\theta_j||^{1+\epsilon'}} < \infty,$$

and this proves the lemma.

As immediate consequences of Lemma I we have, for almost all Θ,

$$\sum_1^N \frac{1}{m \prod_1^k |\sin m\pi\theta_j|} = O(\log^{k+1+\epsilon} N); \tag{8.2}$$

$$\sum_1^\infty \frac{1}{m^{1+r} \prod_1^k |\sin m\pi\theta_j|} = O(1) \qquad (r > 0). \tag{8.3}$$

(5.2) of Theorem II follows from Theorem III and (8.3).

To prove (5.1) we write

$$T_1^{(s)}(\eta) = \int^\eta T_0^{(s)}(\eta)\, d\eta$$

$$= \frac{1}{2^s \pi^2}\left(\omega_j \sum_1^\infty \frac{\cos\left(\frac{2\lambda_j \pi}{\omega_j}\left(\eta - \frac{1}{2}\sum_1^s \omega\right) - \frac{1}{2}(s+1)\pi\right)}{\lambda_j^2 \prod_{k\neq j} \sin\frac{\lambda_j \omega_k \pi}{\omega_j}}\right)^* \tag{8.4}$$

$$= \frac{1}{2^s \pi^2}(\omega_j \sum_{\lambda_j \delta < 1})^* + \frac{1}{2^s \pi^2}(\omega_j \sum_{\lambda_j \delta \geq 1})^* = I_1 + I_2,$$

and let $\Delta f = f(x+\delta) - f(x)$. Then

$$\Delta I_1 = O\left(\delta \sum_{m\delta < 1} \frac{1}{m \prod_{k\neq j} \left|\sin\frac{m\pi\omega_k}{\omega_j}\right|}\right) = O\left(\delta \log^{s+\epsilon}\frac{1}{\delta}\right) \tag{8.5}$$

by (8.2). Next, summing by parts,

$$I_2 = O\left(\delta \log^{s+\epsilon} \frac{1}{\delta}\right), \tag{8.6}$$

and so, combining (8.5) and (8.6),

$$\Delta T_1^{(s)}(\eta) = O\left(\delta \log^{s+\epsilon} \frac{1}{\delta}\right). \tag{8.7}$$

Finally since $N_0^{(s)}(\eta)$ is non-decreasing, we have

$$\delta N_0^{(s)}(\eta) \leq \Delta N_1^{(s)}(\eta) \leq \delta N_0^{(s)}(\eta + \delta). \tag{8.8}$$

Also

$$\Delta\zeta_s(-1, \eta) = \delta\zeta_s(0, \eta) + O(\delta^2 \eta^{s-1}). \tag{8.9}$$

Hence by (8.7), (8.8), and (8.9):

$$\begin{aligned} N_0^{(s)}(\eta) \leq \frac{\Delta N_1^{(s)}}{\delta} &= (-1)^s \zeta_s(0, \eta) + O(\delta\eta^{s-1}) + O\left(\frac{\Delta T_1^{(s)}}{\delta}\right) \\ &= (-1)^s \zeta_s(0, \eta) + O(\delta\eta^{s-1}) + O\left(\log^{s+\epsilon} \frac{1}{\delta}\right), \end{aligned} \tag{8.10}$$

and the opposite inequality comes similarly. (5.1) follows by choosing $\delta = 1/\eta^{s-1}$.

Easy extensions of the latter half of the proof show in particular that the study of $T_r^{(s)}(\eta)$ is reduced to the consideration of the partial sums

$$\sum_1^N \frac{\cos\left(\frac{2\lambda_j \pi}{\omega_j}(\eta - \frac{1}{2}\sum \omega) - \frac{1}{2}(r+s)\pi\right)}{\lambda_j^{1+r} \prod_{k \neq j} \sin \frac{\lambda_j \omega_k \pi}{\omega_j}}. \tag{8.11}$$

9. The above analysis (though superficial) raises several interesting questions; for example:

(i) What is the smallest possible order of magnitude of $T_0^{(s)}(\eta)$ for all $\Omega = (\omega_1, \omega_2, \cdots, \omega_s)$? It is known that $|T_0^{(s)}(\eta)| > K \log \eta$ for a sequence of η tending to infinity and all Ω.

(ii) Do there exist points $\Theta = (\theta_1, \theta_2, \cdots, \theta_k)$ such that

$$\prod_1^k |\sin m\pi\theta_j| \geq \frac{K}{m} \qquad (m \geq 1)$$

when $k > 1$? If not, what is the function $\varphi_k(m)$ of slowest growth for which the inequality

$$\prod_1^k |\sin m\pi\theta_j| \geq \frac{K}{m\varphi_k(m)}$$

is true for at least one Θ and all $m \geq 1$?

(iii) Are there Θ for which, uniformly in η,

$$\left| \sum_1^N \frac{e^{im\pi\eta}}{m \prod_1^k \sin m\pi\theta_j} \right| = O(\log^k N)$$

when $k > 1$? Are there Θ such that

$$\left| \sum_1^N \frac{1}{m \prod_1^k \sin m\pi\theta_j} \right| = O(1)$$

when $k > 1$?

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ON THE THEORY OF ELASTIC VIBRATIONS IN PLATES AND SHELLS

By Paul S. Epstein

1. **Introduction.** The classical theory of deformations and oscillations in thin elastic shells was worked out by Poisson (1) and Kirchhoff (2) for plane parallel plates and extended by Love (3) to curved shells. The two basic assumptions of this theory were most clearly stated by Kirchhoff and by Lamb. In Lamb's words they are as follows.

(A) The state of strain is such that, any arbitrary deformation having been given to the middle surface, a line of particles originally normal to this surface remains straight and normal to it in the strained condition.

(B) The elongation in the direction of the normal is adjusted so as to give zero traction across any small area parallel to the middle surface.

While these assumptions make the problem definite, it cannot be said that they are physically convincing. Especially, the justification of the thesis (B) is by no means obvious. On the contrary, the feeling is widespread that these postulates introduce arbitrary constraints which do not lie in the nature of the phenomenon.

The fundamental equations of the classical shell theory, based on these assumptions, were given by its authors to terms of the second order in the half thickness h of the shell or plate. They consist of a part independent of h (zero order terms) and of another part, proportional to h^2 (second order terms). The theory does not provide any method for the calculation of terms of higher order than the second. While the inaccuracies, alleged above as contained in the basic assumptions, could hardly have any influence on the zero order terms, possibly, they may seriously affect the second order terms whose validity is, thus, rendered questionable. Even though the second order terms, admittedly, are less important, it would be well to know to what extent they are correctly represented by the classical analysis. It seems, therefore, highly desirable to find for the theory of shells a new foundation closer to the actual conditions which would permit of a systematic derivation of the terms of second and of higher orders without arbitrary assumptions or other ambiguities.

Such a new approach is offered in the following sections. It consists in starting from the conditions in a *thick* shell, regarded as a three dimensional continuum, and in carrying out the transition to the limit of a very small thickness, while retaining during this process the validity of the three dimensional field equations and of the border conditions at the free surfaces. The details of the method are explained in section 2.

In the present paper the method is carried through for shells of any shape only in the first approximation. As anticipated, the zero order terms are identical with those of the classical theory (sections 3 and 4). The second order terms are calculated for the special case of circularly cylindric shells (of constant thickness) which includes plane parallel plates as a limiting case. These terms are materially different from those found in Love's theory (section 6), a fact

which has also a bearing on the border conditions at the edges of the shell (section 7).

The theory is applied to flexural and torsional vibrations of a cylindric shell as examples (section 8). While the old frequency expressions are confirmed in the flexural case, the results are different from those of Love's theory in the case of torsional oscillations. This difference, though numerically slight, is theoretically highly significant. The conditions for torsional deformations are so simple that this problem can be rigorously solved by Saint Venant's method. The formula obtained for the frequency of torsional vibrations from the exact treatment agrees with that resulting from our theory[1] but disagrees with that deduced from Love's.

It is improbable that our equations can be directly derived from any simple variation principle. Although this fact is a disadvantage in practice, it seems to be unavoidable and to lie in the nature of the problem.

2. **Explanation of the method.** We describe the space within a thick shell u, v, w and we let the two free surfaces of the shell belong to the family $u = \text{const}$, respectively, corresponding to the parameter values $u = u_1$ and $u = u_2$. We assume, moreover, that the vibrational state of the shell is periodic with the frequency ω involving the time factor $\exp(-i\omega t)$. The elastic displacement $\mathbf{s}$ is, then, subject to the usual field equations

$$\nabla\times\nabla\times\mathbf{s} - (\nu + 2)\nabla(\nabla\cdot\mathbf{s}) - \kappa^2\mathbf{s} = 0, \tag{1}$$

$$\nu = \lambda/\mu, \qquad \kappa^2 = \omega^2\rho/\mu, \tag{2}$$

where ρ is the density of the shell material and λ and μ are its elastic constants. On the other hand, the border conditions at the free surfaces consist in the vanishing of the stress components

$$\begin{aligned} p_{uu}(u_1) &= p_{uv}(u_1) = p_{uw}(u_1) = 0, \\ p_{uu}(u_2) &= p_{uv}(u_2) = p_{uw}(u_2) = 0. \end{aligned} \tag{3}$$

If the line element dl is expressed in our coordinates as

$$dl^2 = U^2\,du^2 + V^2\,dv^2 + W^2\,dw^2,$$

the stresses as expressed in terms of the displacement components have the form

$$\begin{aligned} \frac{1}{\mu}p_{uu} &= \nu\,\nabla\cdot\mathbf{s} + 2\sigma_u, \qquad \sigma_u = \frac{\partial}{\partial u}\left(\frac{s_u}{U}\right) + \mathbf{s}\cdot\nabla\log U, \\ \frac{1}{\mu}p_{uv} &= \frac{V}{U}\frac{\partial}{\partial\mu}\left(\frac{s_v}{V}\right) + \frac{U}{V}\frac{\partial}{\partial\mu}\left(\frac{s_u}{U}\right), \end{aligned} \tag{4}$$

and so on, by cyclic substitution.

The problem of the thick shell is completely determined by the eqs. (1) and (2). In order to go to the limit of the thin shell, we designate the parameter

[1] I am indebted to my colleague Professor Th. Von Karman for drawing my attention to this fact.

characterizing its middle surface by $u = a$, and the parameters at the free surfaces by $u_1 = a + h$, $u_2 = a - h$. Moreover, we assume that, for small values of h, the equations (3) can be represented by the Taylor expansion

$$p_{uu}(a \pm h) = p_{uu}(a) \pm hp'_{uu}(a) + \tfrac{1}{2}h^2p''_{uu}(a) \pm \tfrac{1}{6}h^3p'''_{uu}(a) + \cdots = 0,$$

etc., where we indicate by the accent a differentiation with respect to u. Adding or substracting the equations with the upper and the lower signs,

$$p_{uu}(a) + \tfrac{1}{2}h^2p''_{uu}(a) + \cdots = 0, \tag{5}$$

$$p_{uu}(a) + \tfrac{1}{6}h^2p'''_{uu}(a) + \cdots = 0, \tag{6}$$

and two more pairs of equations in which the second subscripts u are replaced by v and w, respectively. It is well to point out that the assumption (B) of the Poisson-Love theory does not agree with the conditions (5) and (6). In fact, this assumption postulates *vanishing normal traction*, i.e., $p_{uu} = 0$ (and consequently $p'_{uu} = 0$) throughout the whole thickness of the shell. Contrary to this the conditions (5), (6) imply that p_{uu} and p'_{uu} vanish only at the free surfaces but be different from zero at the middle surface.

The eqs. (1) are three in number and of the second order. They can be resolved with respect to the three partials s''_u, s''_v, s''_w for which expressions involving only partials of first and zero orders, with respect to u, are thus obtained. Repeated differentiation of the eqs. (1) gives the possibility of expressing also higher derivatives of the components of $\mathbf{s}$ in terms of the same partials (of first and zero orders with respect to u but of any order with respect to v and w). Substituting these expressions into the eqs. (5) and (6), we reduce them to relations between partials (with respect to u) of zero and first orders only. The next step is to use the eq. (5) with its two analogs to express s'_u, s'_v, s'_w in terms of quantities altogether independent of u. Substituting the expressions so obtained into the eq. (6) and its analogs, we obtain three relations involving only the coordinates v, w. These are the equations for the thin shell which constitute the object of our quest.

On principle, it is thus possible to obtain the equations of vibrations in a curved shell to any degree of approximation in h. In practice, however, the procedure is laborious. In the present paper the method is carried through generally only for terms of zero order in h. Second order terms in h are calculated only for the special case of cylindric shells. Even with this restriction the calculation of the second order terms takes up by far the larger part of the space.

3. **First approximation.** The procedure outlined in the preceding section can be greatly simplified in practice by the use of the familiar form of the field equations written in terms of the stresses

$$\mu\kappa^2 s_u - \frac{1}{UVW}\left\{\frac{\partial}{\partial u}(VWp_{uu}) + \frac{\partial}{\partial v}(WUp_{uv}) + \frac{\partial}{\partial w}(UVp_{uw}) + W\frac{\partial U}{\partial v}p_{uv} + V\frac{\partial U}{\partial w}p_{uw} - W\frac{\partial V}{\partial u}p_{vv} - V\frac{\partial W}{\partial u}p_{vw}\right\} = 0, \tag{7}$$

and two other equations resulting from this one by cyclic substitution.

It is our goal to replace the stress components by others which are defined exclusively with respect to the shell, that is, only with respect to the coordinates v, w varying along the shell. The components containing the subscript u can be eliminated (in the first approximation) by means of the relations (5) and (6). On the other hand, the component p_{vw} need not be replaced as it is defined independently of the u direction. We shall indicate this by writing it with a capital letter

$$p_{vw} = P_{vw}.$$

There remain, however, still the components p_{vv}, p_{ww} which depend on $\nabla \cdot \mathbf{s}$, according to eq. (4), and, therefore, contain a partial derivative with respect to u. It is possible to reduce the disturbing part of these components to p_{uu} in view of the relation

$$\nabla \cdot \mathbf{s} = \sigma_u + \sigma_v + \sigma_w. \tag{8}$$

We introduce the notation

$$A = 2(\nu + 1)/(\nu + 2), \tag{9}$$

whence $A - 1 = \nu/(\nu + 2)$, $2 - A = 2/(\nu + 2)$, and define the two dimensional stress components as

$$P_{vv} = 2\mu\,[A\sigma_v + (A - 1)\sigma_u], \qquad P_{ww} = 2\mu\,[(A - 1)\sigma_v + A\sigma_w]. \tag{10}$$

In this way we obtain the relations

$$p_{vv} = (A - 1)p_{uu} + P_{vv}, \qquad p_{ww} = (A - 1)p_{uu} + P_{ww}. \tag{11}$$

Eliminating p_{uu}, p_{uv}, p_{uw} with the help of (5) and (6), we find the system

$$\left.\begin{aligned}
\mu\kappa^2 s_u &= \frac{1}{UVW}\left\{W\frac{\partial V}{\partial u}P_{vv} + V\frac{\partial W}{\partial u}P_{ww}\right\} + \mu h^2 D_1,\\
\mu\kappa^2 s_v &= -\frac{1}{UVW}\left\{\frac{\partial}{\partial v}(WUP_{vv}) + \frac{\partial}{\partial w}(UVP_{vw})\right.\\
&\qquad \left. + U\frac{\partial V}{\partial w}P_{vw} - U\frac{\partial W}{\partial v}P_{ww}\right\} + \mu h^2 D_2,\\
\mu\kappa^2 s_w &= -\frac{1}{UVW}\left\{\frac{\partial}{\partial v}(WUP_{vw}) + \frac{\partial}{\partial w}(UVP_{ww})\right.\\
&\qquad \left. + U\frac{\partial W}{\partial v}P_{vw} - U\frac{\partial V}{\partial w}P_{vv}\right\} + \mu h^2 D_3.
\end{aligned}\right\} \tag{12}$$

$$\left.\begin{aligned}
2\mu D_1 &= \frac{1}{UVW}\left\{(2 - A)\frac{\partial(VW)}{\partial u}p''_{uu} + \frac{\partial}{\partial v}(WUp''_{uv}) + \frac{\partial}{\partial w}(UVp''_{uw})\right.\\
&\qquad \left. + W\frac{\partial U}{\partial v}p''_{vv} + V\frac{\partial U}{\partial w}p''_{ww} + \tfrac{1}{3}VWp'''_{uu}\right\},\\
2\mu D_2 &= \frac{1}{UVW}\left\{(A - 1)\frac{\partial}{\partial v}(UWp''_{uu}) - \frac{\partial(UW)}{\partial v}p''_{uu}\right.\\
&\qquad \left. + \left(2W\frac{\partial V}{\partial u} + V\frac{\partial W}{\partial u}\right)p''_{uv} + \tfrac{1}{3}VWp'''_{uv}\right\},
\end{aligned}\right\} \tag{13}$$

$$2\mu D_3 = \frac{1}{UVW}\left\{(A-1)\frac{\partial}{\partial w}(UVp''_{uu}) - \frac{\partial(UV)}{\partial w}p''_{uu} + \left(2V\frac{\partial W}{\partial u} + W\frac{\partial V}{\partial u}\right)p''_{uw} + \tfrac{1}{2}VWp'''_{uw}\right\}.$$

When the terms with h^2 in (12) are neglected these equations represent the first approximation of the theory of elastic shells. They agree, in fact, with the equations given by Lamb on the basis of Love's theory being a slight generalization of those equations.

4. **First approximation for a cylindric shell.** In this particular case the coordinates that must be used are the cylindric mesh system r, φ, z. In writing the partials with respect to φ and z we shall use the simplified symbolic notation

$$\partial/\partial\varphi = im, \qquad \partial/\partial z = i\alpha = i\gamma/a, \tag{14}$$

a being the radius of the middle surface.

The symbols m and α would represent numbers if the dependence of $\mathfrak{s}$ on φ and z were given by $\exp i(m\varphi + \alpha z)$. On the other hand, we shall represent the partial differentiation with respect to r by an accent

$$\partial\mathfrak{s}/\partial r = \mathfrak{s}'. \tag{15}$$

In these notations the stress components take the form

$$\left.\begin{aligned} P_{\varphi z}/\mu &= i\alpha s_\varphi + \frac{im}{r}s_z, \\ P_{\varphi\varphi}/\mu &= \frac{2A}{r}(s_r + ims_\varphi) + 2(A-1)i\alpha s_z, \\ P_{zz}/\mu &= 2\left(\frac{A-1}{r}\right)(s_r + ims_\varphi) + 2Ai\alpha s_z. \end{aligned}\right\} \tag{16}$$

Applying this to the first approximation of the equations (12), we bear in mind that, for $h = 0$, we have $r = a$. Using, moreover, the abbreviation

$$\Omega = \kappa^2a^2 = \rho a^2\omega^2/\mu, \tag{17}$$

we find the explicit eqs.

$$\left.\begin{aligned} \Omega s_r &= 2[As_r + Aims_\varphi + (A-1)i\gamma s_z], \\ \Omega s_\varphi &= -2Aims_r + (2Am^2 + \gamma^2)s_\varphi + (2A-1)m\gamma s_z, \\ \Omega s_z &= -2(A-1)i\gamma s_r + (2A-1)m\gamma s_\varphi + (m^2 + 2A\gamma^2)s_z. \end{aligned}\right\} \tag{18}$$

5. **Auxiliary expressions.** We go now to the calculation of the terms of second order in (12). As these terms contain the factor h^2, the quantities D need be calculated only to terms of zero order. That means we have to set $h = 0$

in the conditions (5) and (6). Because of the definitions (4), the conditions (5) yield, in the case of a cylindric shell, the relations (for $r = a$)

$$\left.\begin{aligned} s_r' &= -\frac{A-1}{a}(s_r + ims_\varphi + i\gamma s_z), \\ s_\varphi' &= \frac{1}{a}(-ims_r + s_\varphi), \\ s_z' &= -\frac{1}{a} i\gamma s_r . \end{aligned}\right\} \tag{19}$$

Similarly, the conditions (6) give when (19) is taken into account

$$\left.\begin{aligned} s_r'' &= \frac{A-1}{a^2}[(A - m^2 - \gamma^2)s_r + (A-1)ims_\varphi + (A-1)i\gamma s_z], \\ s_\varphi'' &= \frac{A-1}{a^2}[ims_r - m^2 s_\varphi - m\gamma s_z], \\ s_z'' &= \frac{A-1}{a^2}[i\gamma s_r - m\gamma s_\varphi - \gamma^2 s_z]. \end{aligned}\right\} \tag{20}$$

Taking into account the relations (11), the field equations (7) acquire in cylindric coordinates the form

$$\left.\begin{aligned} p_{rr}' &= \frac{A-1}{r} p_{rr} - \frac{im}{r} p_{r\varphi} - i\alpha p_{rz} + \frac{1}{r} P_{\varphi\varphi} - \mu\kappa^2 s_r, \\ p_{r\varphi}' &= (1-A)\frac{im}{r} p_{rr} - \frac{2}{r} p_{r\varphi} - \frac{im}{r} P_{\varphi\varphi} - i\alpha P_{\varphi z} - \mu\kappa^2 s_\varphi, \\ p_{rz}' &= (1-A)\, i\alpha p_{rr} \frac{1}{r} p_{rz} - \frac{im}{r} P_{\varphi z} - i\alpha P_{zz} - \mu\kappa^2 s_z . \end{aligned}\right\} \tag{21}$$

The partials p_{rr}'', p_{rr}''', etc., are obtained by differentiating once or twice with respect to r. We list here the formulas for several functions which occur in the expressions of these partials.

From (19) and (18) we find

$$\kappa^2 s_r' = -2\frac{A-2}{a^3}\{[A + Am^2 + (A-1)\gamma^2]s_r + Aim(1 + m^2 + \gamma^2)s_\varphi + [A - 1 + A(m^2 + \gamma^2)]i\gamma s_z\},$$

$$\kappa^2 s_\varphi' = \frac{1}{a^3}\{-4Aims_r + (4Am^2 + \gamma^2)s_\varphi + (4A-3)m\gamma s_z\},$$

$$\kappa^2 s_z' = \frac{2}{a^3}\{-Ai\gamma s_r + Am\gamma s_\varphi + (A-1)\gamma^2 s_z\}.$$

From (20) and (18)

$$\kappa^2 s_r'' = 2\frac{A-1}{a^4}\{[A^2 + A(A-2)m^2 + (A^2 - 3A + 1)\gamma^2]s_r$$
$$+ A[A + (A-2)(m^2+\gamma^2)]ims_\varphi$$
$$+ (A-1)[A + (A-1)(m^2+\gamma^2)]i\gamma s_z\},$$

$$\kappa^2 s_\varphi'' = 2\frac{A-1}{a^4}\{[A + Am^2 + (A-1)\gamma^2]ims_r - A(1+m^2+\gamma^2)m^2 s_\varphi$$
$$- [A - 1 + A(m^2+\gamma^2)]m\gamma s_z\},$$

$$\kappa^2 s_z'' = 2\frac{A-1}{a^4}\{[A + Am^2 + (A-1)\gamma^2]i\gamma s_r - A(1+m^2+\gamma^2)m\gamma s_\varphi$$
$$- [A - 1 + A(m^2+\gamma^2)]\gamma^2 s_z\}.$$

From (16), (19), and (20)

$$P'_{\varphi\varphi} = -\frac{2\mu}{a^2}\{[A(A-m^2) - (A-1)\gamma^2]s_r + A(A-1)(ims_\varphi + i\gamma s_z)\},$$

$$P''_{\varphi\varphi} = \frac{2\mu}{a^3}\{[A^2(A+1) - 2A^2m^2 - (A-1)(2A-1)\gamma^2]s_r$$
$$+ (A-1)[A(A+1) - Am^2 - (A-1)\gamma^2](ims_\varphi + i\gamma s_z)\},$$

$$P'_{zz} = -\frac{2\mu}{a^2}\{[(A-1)(A-m^2) - A\gamma^2]s_r + (A-1)^2(ims_\varphi + i\gamma s_z)\},$$

$$P''_{zz} = \frac{2\mu}{a^3}(A-1)\{[A(A+1) - 2Am^2 - (2A-1)\gamma^2]s_r$$
$$+ [A^2 - 1 - (A-1)m^2 - A\gamma^2](ims_\varphi + i\gamma s_z)\},$$

$$P'_{\varphi z} = \frac{\mu}{a^2}(2m\gamma s_r + i\gamma s_\varphi - ims_z),$$

$$P''_{\varphi z} = -\frac{2\mu}{a^3}\{Am\gamma s_r + (A-1)im^2\gamma s_\varphi + [(A-1)\gamma^2 - 1]ims_z\}.$$

The second derivatives can now be obtained without any trouble

$$p_{rr}'' = \frac{2\mu}{a^3}\{A[-2 + Am^2 + (A-1)\gamma^2]s_r$$
$$- A[1 - (A-1)(m^2+\gamma^2)]ims_\varphi - (A-1)[2 - A(m^2+\gamma^2)i\gamma s_z\},$$

$$p''_{r\varphi} = \frac{2\mu}{a^3}\{A(A+3-m^2-\gamma^2)ims_r - A(A+2)m^2 s_\varphi - (A^2+2A-2)m\gamma s_z\},$$

$$p''_{rz} = \frac{2\mu}{a^3}\{A(A-m^2-\gamma^2)i\gamma s_r - (A^2-A+1)m\gamma s_\varphi - [m^2+A(A-1)\gamma^2]s_z\}.$$

6. Second approximation of fundamental equations for cylindric shells. The third derivatives of the stresses are obtained by twice differentiating the eqs. (21)

$$p'''_{rr} = \frac{1}{a}[(A-2)p''_{rr} - imp''_{r\varphi} - i\gamma p''_{rz} + P''_{\varphi\varphi}] - \frac{1}{a^2}[2(P'_{\varphi\varphi} - a^{-1}P_{\varphi\varphi}) + \mu\Omega s''_r],$$

$$p'''_{r\varphi} = \frac{1}{a}[(1-A)\,imp''_{rr} - 2p''_{r\varphi} - imP''_{\varphi\varphi} - i\gamma P''_{\varphi z}] + \frac{1}{a^2}[2im(P'_{\varphi\varphi} - a^{-1}P_{\varphi\varphi}) - \mu\Omega s''_\varphi],$$

$$p'''_{rz} = \frac{1}{a}[(1-A)i\gamma p''_{rr} - p''_{rz} - imP''_{\varphi z} - i\gamma P''_{zz}] + \frac{1}{a^2}[2im(P'_{\varphi z} - a^{-1}P_{\varphi z}) - \mu\Omega s''_z].$$

Substituting this into the eqs. (13), we find

$$D_1 = \frac{1}{3a}\left\{(2-A)p''_{rr} + imp''_{r\varphi} + i\gamma p''_{rz} + \frac{1}{2}P''_{\varphi\varphi} - \frac{1}{a}\left(P'_{\varphi\varphi} - a^{-1}P_{\varphi\varphi} + \frac{1}{2}\mu\Omega s''_r\right)\right\},$$

$$D_2 = \frac{1}{3a}\left\{(A-1)imp''_{rr} + 2p''_{r\varphi} - \frac{1}{2}imP''_{\varphi\varphi} - \frac{1}{2}i\gamma P''_{\varphi z} + \frac{im}{a}(P'_{\varphi\varphi} - a^{-1}P_{\varphi\varphi}) - \frac{\mu}{2a}\Omega s''_\varphi\right\},$$

$$D_3 = \frac{1}{3a}\left\{(A-1)i\gamma p''_{rr} + p''_{rz} - \frac{1}{2}imP''_{\varphi z} - \frac{1}{2}i\gamma P''_{zz} + \frac{im}{a}(P'_{\varphi z} - a^{-1}P_{\varphi z}) - \frac{\mu}{2a}\Omega s''_z\right\}.$$

All the terms of these expressions are listed in the preceding section. Hence, we can obtain the explicit formulas for the second order terms of our fundamental equations. To the right sides of the eqs. (18) must be added the following expressions

$$
\left.
\begin{aligned}
h^2a^2D_1 = \frac{h^2}{3a^2}\{&[2A(4A-3) - A(3A^2-3A+10)m^2 \\
&- (3A^3-6A^2+7A-2)\gamma^2 + 2A(m^2+\gamma^2)^2]s_r \\
&+ [5A(A-1) - 3A(A^2-2A+3)m^2 \\
&- (3A^3-6A^2+2A+3)\gamma^2]ims_\varphi \\
&+ [2(A-1)(4A-3) - (3A^3-6A^2+10A-3)m^2 \\
&+ A(A-1)^2\gamma^2]i\gamma s_z\}, \\
h^2a^2D_2 = \frac{h^2}{3a^2}\{&[-A(A^2+4A-15) + A(2A^2-A-1)m^2 \\
&+ (2A^3-3A^2-2)\gamma^2]ims_r \\
&+ A[A^2+A-12-2(A-1)^2(m^2+\gamma^2)]m^2s_\varphi \\
&+ [A^3+3A^2-19A+12-2A(A-1)^2(m^2+\gamma^2)]m\gamma s_z\}, \\
h^2a^2D_3 = \frac{h^2}{3a^2}\{&[-A(A^2+3A-6) + (2A^3-A^2-2A-2)m^2 \\
&+ A(2A^2-3A-1)\gamma^2]i\gamma s_r \\
&+ [A^3-2A-1-2A(A-1)^2(m^2+\gamma^2)]m\gamma s_\varphi \\
&+ [m^2 + (A-1)(A^2+3A-6)\gamma^2 \\
&- 2A(A-1)^2(m^2+\gamma^2)\gamma^2]s_z\}.
\end{aligned}
\right\} \quad (22)
$$

These expressions are materially different from those obtaining in Love's theory.

The equations are linear and have constant coefficients. Therefore, they admit of particular integrals of the type

$$
\begin{aligned}
s_r &= \exp(im\varphi)\sin \alpha z\, R, \\
s_\varphi &= \exp(im\varphi)\sin \alpha z\, \Phi, \\
s_z &= \exp(im\varphi)\cos \alpha z\, Z,
\end{aligned} \quad (23)
$$

where R, Φ, Z signify constants.

7. **Border conditions.** The border conditions at the edges of the shell are immediately determined by those obtaining in the three dimensional continuum (thick shell) whose limiting case the shell represents (compare section 2). If the edge of the thick shell coincides with a surface $l = \text{const}$, the border conditions at this surface are

(A) $s_u = s_v = s_w = 0$, for clamped edges,

(B) $p_{lu} = p_{lv} = p_{lw} = 0$, for free edges,

(C) in addition *mixed conditions* ("supported edges") may be considered.

As our eqs. (12) of the first approximation are the same as in Love's theory,

nothing new can be said about them. We shall only state for what kind of border conditions the solution can be represented by the simple integrals (23).

It is well known that the conditions (A), for clamped edges, cannot be satisfied by solutions of the type (23). Turning to the conditions (13), for free edges, we notice that in this approximation one of them, namely $p_{lu} = 0$, is already satisfied because of the fundamental assumptions (5) and (6) of our theory. There remain only the conditions $p_{lv} = p_{lw} = 0$ to satisfy. In the particular case when the vibrating system is the section of a cylinder limited by two planes $z = \text{const}$, the conditions are, thus, $p_{zz} = p_{zz} = 0$, or

$$\frac{1}{a}\frac{\partial s_z}{\partial \varphi} + \frac{\partial s_\varphi}{\partial z} = 0\,, \qquad \frac{\partial s_z}{\partial z} = 0\,. \tag{24}$$

In general, also these conditions cannot be satisfied by simple expressions of the type (23). However, the special cases $m = 0$ (axial symmetry) and $\alpha = 0$ (independence from the longitudinal coordinates) form exceptions. In these cases, one of the two conditions can be made to be satisfied identically while the other can be fulfilled by a proper choice of the constant α.

Some of the mixed border conditions (C) can be also adjusted so as to make solutions of the type (23) appropriate.

If we turn from the first approximation to the second, the conditions are more complicated. In some of the cases, where the first approximation admits of solutions of the simple type, this is no longer true in the second approximation. The complication arises from the fact that the condition $p_{rz} = 0$ which must be imposed at the free edge, is no longer automatically guaranteed since, in this approximation, the fundamental eqs. (5) require $p_{rz} = -\frac{1}{2}h^2 p''_{rz}$. In the two cases, mentioned above as satisfying the conditions (24) the simple form of the solutions cannot be retained unless p''_{rz} happens to be equal to zero. This is the case only under the following additional restrictions.

(1) For $\alpha = 0$, when $s_z = 0$, while s_r and s_φ do not vanish. This includes the case of the so called *flexural oscillations*.

(2) For $m = 0$, when $s_r = s_z = 0$, while s_φ does not vanish. This is the case of *torsional oscillations*.

In order to have simple illustrations for our results, we shall apply them to the two cases of flexural and torsional vibrations.

8. **Flexural and torsional oscillations.** The case of flexural vibrations ($\alpha = 0$, $s_z = 0$) is useful as a check upon the correctness of our calculations. For reasons of conservation of momentum it is evident, that the oscillation with $m = 1$ cannot exist. Hence, the expression of Ω (proportional to ω^2) must contain the factor $m^2 - 1$. The general structure of the equations makes it, moreover, certain that this factor will occur in the combination $m^2(m^2 - 1)/(m^2 + 1)$.

Substituting $\gamma = 0$, $s_z = 0$ into the eqs. (18) and (22) and writing $\delta = \frac{1}{3}(h/a)^2$, we find

$$\left.\begin{aligned} (\Omega/A - 2 - \delta D_{11})s_r - im(2 + \delta D_{12})s_\varphi = 0\,, \\ im(2 - \delta D_{21})s_r + (\Omega/A - 2m^2 + \delta D_{22})s_\varphi = 0\,. \end{aligned}\right\} \tag{24'}$$

$$D_{11} = 2(4A - 3) - (3A^2 - 3A + 10)m^2 + 2m^4,$$
$$D_{12} = 5(A - 1) - 3(A^2 - 2A + 3)m^2,$$
$$D_{21} = -(A^2 + 4A - 15) + (2A^2 - A - 1)m^2,$$
$$D_{22} = -(A^2 + A - 12)m^2 + 2(A^2 - 2A + 1)m^4.$$

The characteristic frequency is obtained by equating to zero the determinant of the system (24'). The flexural oscillations are those for whose frequencies the first approximation is $\Omega = 0$. The second approximation yields

$$\Omega(m^2 + 1) = \delta[m^2(D_{11} - D_{12} + D_{21}) - D_{22}],$$

or

$$\Omega = 2\delta A m^2(m^2 - 1)^2/(m^2 + 1).$$

The expression is, in fact, of the expected type whence we may conclude that, at least, the terms independent of γ in our expressions (22) are free of errors of calculation. Because of the definition (10) of Ω we find for the frequency

$$\omega = \left(\frac{2\mu A}{3\rho}\right)^{\frac{1}{2}} \frac{h}{a^2} \frac{m(m^2 - 1)}{(m^2 + 1)^{\frac{1}{2}}}. \tag{25}$$

This expression is identical with that worked out by Rayleigh on the basis of Love's theory. The flexural oscillations are, in fact, not suitable to bring out differences. For the reasons stated in the beginning of this section any theory satisfying the law of conservation of momentum must yield for them the same, or nearly the same, frequency.

It is otherwise with the *torsional vibrations* which are obtained from (18) and (22) by putting $m = 0$, $s_r = s_z = 0$. There follows simply

$$\Omega = \gamma^2.$$

The conditions (24) are reduced, in this case, to $\partial s_\varphi/\partial z = 0$ or $\cos \alpha z = 0$, according to (23). If the edges of the cylindric section lie at $z = -l/2$ and $z = l/2$, this condition is fulfilled when $\alpha = n\pi/l$, where, n is an integer. Hence, the frequency is expressed by

$$\omega = \left(\frac{\mu}{\rho}\right)^{\frac{1}{2}} \frac{\pi n}{l}. \tag{26}$$

This expression does not contain any second order terms, as D_2 vanishes for $m = 0$, according to the eqs. (22). It is different in Love's theory where the result does depend on second order terms, being

$$\omega = (\mu/\rho)^{\frac{1}{2}}(1 + h^2/6a^2)\pi n/l.$$

As was stated in section 1, the case of torsional oscillations can be rigorously treated by Saint Venant's method. The exact expression of the frequency for a solid or hollow cylinder of any radius (or radii) is identical with our formula (26).

9. Limiting case of the plane parallel plate. A plane parallel plate may be regarded as a cylinder of infinite radius. Therefore, we substitute into the eqs. (18) and (22), $\Omega = \kappa^2 a^2$, changing at the same time the symbols r into x and $a\varphi$ into y and, correspondingly, replacing

$$im = a\partial/\partial y, \qquad i\gamma = a\partial/\partial z.$$

When the transition to the limit $a = \infty$ is carried out, many terms vanish. Writing, for short, $f = \frac{2}{3}(h/a)^2 A(A-1)^2$, we find the system

$$\kappa^2 s_x - \frac{2}{3} h^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 s_x = 0,$$

$$\kappa^2 s_y + \left[2A \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + f\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\frac{\partial^2}{\partial y^2}\right] s_y + \left[2A - 1 + f\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\right] \frac{\partial^2}{\partial y \partial z} s_z = 0,$$

$$\kappa^2 s_z + \left[2A - 1 + f\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\right] \frac{\partial^2}{\partial y \partial z} s_y + \left[\frac{\partial^2}{\partial y^2} + 2A \frac{\partial^2}{\partial z^2} + f\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\frac{\partial^2}{\partial z^2}\right] s_z = 0.$$

In Love's theory the first equation agrees with ours, but the second and third are different, being completely free of terms of the second order.

A summary of the paper is given at the end of the first section.

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ON SOME APPLICATIONS OF THE TCHEBYCHEFF INEQUALITY FOR DEFINITE INTEGRALS

By J. A. Shohat and A. V. Bushkovitch

Introduction

In many problems of Analysis it is of importance to find upper and lower bounds for definite integrals with greater precision than is afforded by the use of the mean value theorem. It is the purpose of this paper to show how this may be done for an important class of definite integrals with the aid of a comparatively little known inequality due to Tchebycheff. The method is used to obtain improved estimates of the remainder in the Taylor expansion of the exponential, logarithmic, trigonometric and hyperbolic functions, and hence, upper and lower bounds for these functions; as well as to derive some inequalities involving π. Further, using some formulas due to G. Kowalewski[1] as a starting point, new and improved upper and lower bounds for the Trapezoidal and Simpson Rules are obtained.

1. Tchebycheff Inequality for Definite Integrals

This inequality states that if $p(x) \geq 0$, and $f_1(x)$ and $f_2(x)$ are monotonic in the finite or infinite interval (a, b), we have

$$\int_a^b p(x)\,dx \cdot \int_a^b p(x) f_1(x) f_2(x)\,dx \gtrless \int_a^b p(x) f_1(x)\,dx \cdot \int_a^b p(x) f_2(x)\,dx \qquad (1)$$

The upper sign holds if $f_1(x)$ and $f_2(x)$ are both increasing or both decreasing; the lower sign applies if one function is decreasing while the other is increasing.

As a matter of fact, as will be clear from the proof, it is sufficient, for the upper (lower) sign to hold, that the functions $f_1(x)$ and $f_2(x)$ be similarly (non-similarly) ordered, i.e. $[f_1(x_1) - f_1(x_2)][f_2(x_1) - f_2(x_2)] \geq 0\ (\leqq 0)$, $a < x_1, x_2 < b$. The applications of the inequality (1) in what follows will be confined to monotonic functions; if the functions are not monotonic, we assume that it is possible to divide the interval of integration into a finite number of subintervals in each of which the functions involved are monotonic.

Tchebycheff himself does not appear to have published the inequality (1), but merely communicated it verbally to Hermite and other mathematicians. The proof which follows is due to F. Franklin.[2]

Write $\Delta = \int_a^b p(x)\,dx \int_a^b p(x) f_1(x) f_2(x)\,dx - \int_a^b p(x) f_1(x)\,dx \int_a^b p(x) f_2(x)\,dx$

as a double integral:

$$\Delta = \int_a^b \int_a^b p(y) p(x) [f_1(x) f_2(x) - f_1(x) f_2(y)]\,dx\,dy$$

[1] *G. Kowalewski*, Interpolation und Genäherte Quadratur, Leipzig, 1932.

[2] *F. Franklin*, Amer. Jour. of Math., VII, p. 377, 1885. For an extension to Stieltjes integrals cf. J. Shohat, C. R. t.189, p. 618, 1929.

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Interchange x and y and add; there results

$$\Delta = \frac{1}{2}\int_a^b \int_a^b p(y)p(x)[f_1(x) - f_1(y)][f_2(x) - f_2(y)]\,dx\,dy,$$

and from this (1) clearly follows.

An important special case (Franklin, l.c.) is the following: In formula (1) let $f_1(x) = f(x)$, $f_2(x) = 1/f(x)$. We obtain

$$\int_a^b p(x)f(x)\,dx \int_a^b \frac{P(x)}{f(x)}\,dx > \left(\int_a^b p(x)\,dx\right)^2, \tag{2}$$

$$p(x) \geq 0 \text{ in } (a, b).$$

We now pass to applications.

2. Remainder in Taylor's Formula

We have

$$f(b) = f(a) + \frac{b-a}{1}f'(a) + \cdots + \frac{(b-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + R_n \tag{3}$$

$$R_n = \frac{1}{(n-1)!}\int_a^b (b-t)^{n-1}f^{(n)}(t)\,dt$$

Assume $f^{(n)}(x)$ monotonic in (a, b), and apply (1) with $p(x) \equiv 1$. There results

$$R_n < \frac{(b-a)^{n-1}}{n!}[f^{(n-1)}(b) - f^{(n-1)}(a)] \tag{4.1}$$

$$f^{(n)}(x) \text{ increasing in } (a, b)$$

$$R_n > \frac{(b-a)^{n-1}}{n!}[f^{(n-1)}(b) - f^{(n-1)}(a)], \tag{4.2}$$

$$f^{(n)}(x) \text{ decreasing in } (a, b)$$

Note that (4.1) and (4.2) do not involve $f^{(n)}(x)$ explicitly.

Illustrations.

(i) $f(t) = e^t$. Letting $a = 0$, we get

$$e^b = 1 + \frac{b}{1!} + \cdots + \frac{b^{n-1}}{(n-1)!} + R_n$$

(4.1) yields

$$R_n = e^b - \sum_{i=0}^{n-1}\frac{b^i}{i!} < \frac{b^{n-1}}{n!}(e^b - 1) \equiv R_n' \quad (b > 0). \tag{5}$$

Whence[3]

$$e^b < \frac{1 + \frac{b}{1!} + \frac{b^2}{2!} + \cdots + \frac{b^{n-2}}{(n-2)!} + \frac{b^{n-1}}{(n-1)!}\left(1 - \frac{1}{n}\right)}{1 - \frac{b^{n-1}}{n!}} \tag{6}$$

$$(b > 0).$$

Note that the above expression (5) for R_n gives a better estimate than the usual form

$$R_n = \frac{b^n}{n!} e^{\theta_n b} < \frac{b^n}{n!} e^b \equiv R_n'' \qquad (0 < \theta_n < 1)$$

since

$$\frac{R_n''}{R_n'} = b \frac{e^b}{e^b - 1} > 1, \qquad \text{if } b > 0.$$

If $b < 0$, we get, by Taylor's Formula,

$$e^b = 1 + \frac{b}{1!} + \frac{b^2}{2!} + \cdots + \frac{b^{n-1}}{(n-1)!} - \frac{1}{(n-1)!} \int_b^0 (b - t)^{n-1} e^t \, dt$$

$$(b < 0).$$

As t increases from b to 0, $(b - t)^{n-1}$ is a decreasing function, if n is even, and an increasing function if n is odd. Thus

$$-R_n = \frac{1}{(n-1)!} \int_b^0 (b - t)^{n-1} e^t \, dt < \frac{b^{n-1}}{n!} [1 - e^b] \quad (n \text{ even}, b < 0)$$

and we obtain, as above,

$$e^b > \frac{1 + \frac{b}{1!} + \frac{b^2}{2!} + \cdots + \frac{b^{n-1}}{(n-1)!}\left(1 - \frac{1}{n}\right)}{1 - \frac{b^{n-1}}{n}} \quad (b < 0, n \text{ even}) \tag{7.1}$$

$$e^b < \frac{1 + \frac{b}{1!} + \frac{b^2}{2!} + \cdots + \frac{b^{n-1}}{(n-1)!}\left(1 - \frac{1}{n}\right)}{1 - \frac{b^{n-1}}{n!}} \quad (b < 0, n \text{ odd}) \tag{7.2}$$

[3] This should be compared with the inequality furnished by the Maclaurin expansion $e^b > 1 + \frac{b}{1!} + \cdots + \frac{b^{n-1}}{(n-1)!}$. From the same expansion we get the following estimate for the remainder

$$R_n = \frac{b^n}{n!} + \frac{b^{n+1}}{(n+1)!} + \cdots < \frac{b^n}{n!} \frac{1}{1 - \frac{b}{n}}.$$

This inequality, however, holds only for $0 < b < n$.

As a numerical illustration, we take $b = 1$ in (6) and $b = -1$ in (7.2), and obtain the following inequalities for $e = 2.71828\ldots$ Taking $n = 6$, $e < 2.7191$, from (6); taking $n = 7$, $e < 2.7184$, from (6), and $e > 2.7179$, from (7.2).

We make the following remark concerning the remainder R_n in (5). The number θ_n in the formula

$$e^b = 1 + \frac{b}{1!} + \cdots + \frac{b^{n-1}}{(n-1)!} + \frac{b^n}{n!} e^{\theta_n b}$$

approaches 0 as $n \to \infty$; more precisely, $\theta_n = 0(1/n)$. The proof is very simple. Write the same formula with n replaced by $n - 1$; then

$$e^b - 1 - \frac{b}{1!} - \cdots - \frac{b^{n-2}}{(n-2)!} = \frac{b^{n-1}}{(n-1)!} e^{\theta_{n-1} b} = \frac{b^{n-1}}{(n-1)!} + \frac{b^n}{n!} e^{\theta_n b},$$

Whence

$$e^{\theta_{n-1} b} = 1 + \frac{b}{n} e^{\theta_n b} \to 1 \qquad (n \to \infty),$$

$$\theta_{n-1} = \frac{1}{b} \log \left(1 + \frac{b}{n} e^{\theta_n b}\right) = 0(1/n).$$

(ii) $f(t) = \log(1 + t)$. We obtain, by the same method,

$$\log(1+t) < t - \frac{t^2}{2} + \frac{t^3}{3} - \cdots + \frac{t^{n-1}}{n-1} - \frac{t^{n-1}}{n(n-1)} \left[\frac{(1+t)^{n-1} - 1}{(1+t)^{n-1}}\right] \tag{8.1}$$

(n even)

$$\log(1+t) > t - \frac{t^2}{2} + \frac{t^3}{3} - \cdots - \frac{t^{n-1}}{n-1} + \frac{t^{n-1}}{n(n-1)} \left[\frac{(1+t)^{n-1} - 1}{(1+t)^{n-1}}\right] \tag{8.2}$$

(n odd)

In particular, letting $t = 1$, we obtain for $\log 2$ the following inequalities:

$$\log 2 < 1 - \frac{1}{2} + \frac{1}{3} - \cdots - \frac{1}{n-2} + \frac{1}{n-1} - \frac{1}{(n-1)n} \frac{2^{n-1} - 1}{2^{n-1}} \tag{9.1}$$

(n even)

and

$$\log 2 > 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{n-2} - \frac{1}{n-1} + \frac{1}{(n-1)n} \frac{2^{n-1} - 1}{2^{n-1}} \tag{9.2}$$

(n odd)

Numerical Illustration. (9.1) and (9.2) yield the following inequalities for $\log 2 = .69315\cdots$: $n = 5$, $\log 2 > .6302$; $n = 6$, $\log 2 < .7511$; $n = 7$, $\log 2 > .6401$; $n = 8$, $\log 2 < .7419$.

(iii) $f(t) = \sin t, \sinh t$. We take in (1) $a = 0, b = a$. In the case of sint, it is sufficient to consider $n = 4k - 1$ and $n = 4k - 3$, and in the case of sinht to take n odd. We get

$$\sin x < \frac{x - \frac{x^3}{3!} + \cdots + \frac{x^{4k-3}}{(4k-3)!}}{1 + \frac{x^{4k-2}}{(4k-1)!}} \qquad \left(0 > x < \frac{\pi}{2}\right) \quad (10.1)$$

$$\sin x > \frac{x - \frac{x^3}{3!} + \cdots - \frac{x^{4k-5}}{(4k-5)!}}{1 - \frac{x^{4k-4}}{(4k-3)!}} \qquad \left(0 < x < \frac{\pi}{2}\right) \quad (10.2)$$

A similar treatment of sinht yields the inequality[4]

$$\sinh x < \frac{x + \frac{x^3}{3!} + \cdots + \frac{x^{n-2}}{(n-2)!}}{1 - \frac{x^{n-1}}{n!}} \qquad (0 < x < \infty) \quad (11)$$

Here n must be sufficiently large to ensure that $x^{n-1} < n!$

3. Applications of formula (2)

This formula yields many interesting inequalities, as illustrated below. If, in (2), we take $p(x) = 1, f(x) = \sqrt{1 - x^2}, a = 0, b = \frac{1}{2}$, we obtain

$$\int_0^{\frac{1}{2}} \sqrt{1 - x^2}\, dx \cdot \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1 - x^2}} > \frac{1}{4},$$

which yields the inequality

$$\pi > \frac{-3\sqrt{3} + \sqrt{315}}{4} = 3.138 \cdots$$

Next, take in formula (2) $p(x) = 1, f(x) = \sin x, a = \frac{\pi}{4}, b = \frac{\pi}{2}$. This gives

$$\int_{\pi/4}^{\pi/2} \sin x\, dx \cdot \int_{\pi/4}^{\pi/2} \operatorname{csc} x\, dx > \left(\int_{\pi/4}^{\pi/2} dx\right)^2$$

$$\pi < 2^{7/4} \sqrt{\log(1 + \sqrt{2})} = 3.158 \cdots$$

A number of further examples of this type will be found in F. Franklin's (l.c.) paper.

[4] This may be compared with the inequality obtained from the Taylor expansion

$$\sinh x > x + \frac{x^3}{3!} + \cdots + \frac{x^{n-2}}{(n-2)!}.$$

4. Mechanical Quadratures

(i) Trapezoidal Rule. We apply (1) to the formula

$$\int_a^b f(x)\,dx = (b-a)\frac{f(a)+f(b)}{2} - \frac{1}{2}\int_a^b (u-a)(b-u)f''(u)\,du$$

(Kowalewski, l.c., p. 40)

Assuming that $f''(u)$ in monotonic in (a, b) and using (1) successively with $p(u) = (u-a)$ and $(b-u)$, we obtain

$$\int_a^b f(x)\,dx > (b-a)\frac{f(a)+f(b)}{2} - \frac{(b-a)^2}{6}\left[f'(b) - \frac{f(b)-f(a)}{b-a}\right]$$

$$\int_a^b f(x)\,dx < (b-a)\frac{f(a)+f(b)}{2} + \frac{(b-a)^2}{6}\left[f'(a) - \frac{f(b)-f(a)}{b-a}\right], \qquad (12)$$

$f''(x)$ increasing in (a, b).

The inequalities (12) are reversed if $f''(x)$ is decreasing in (a, b).

These bounds are in some cases much closer than those given by Kowalewski under a different assumption, namely that $f''(x)$ does not change sign in (a, b). For example, taking $f(x) = x^4$, $a = 1$, $b = 2$, and letting $I = \int_1^2 x^4\,dx = 6.20$, we find, using (12), the inequalities $5.66 < I < 6.67$, whereas Kowalewski's formulas give here $5.00 < I < 8.50$.

(ii) Simpson's Rule. We start with the formula

$$\int_a^b f(x)\,dx = (b-a)\frac{f(a)+4f(m)+f(b)}{6} + \frac{1}{6}\int_a^m (a-u)^2(m-u)f'''(u)\,du$$
$$+ \frac{1}{6}\int_m^b (b-u)^2(m-u)f'''(u)\,du \qquad \left(m = \frac{a+b}{2}\right) \qquad (13)$$

(Kowalewski, l.c., p. 49).

Assume now that $f'''(u)$ is increasing in (a, b). Applying (1) to the two integrals on the right in (13), with $p(u) = (u-a)^2$ in the first integral and $(b-u)^2$ in the second, we get

$$\int_a^b f(x)\,dx < (b-a)\frac{f(a)+6f(m)+f(b)}{8} + \frac{(b-a)^3}{96}f''(m),$$
$$f'''(x) \text{ increasing in } (a, b) \qquad \left(m = \frac{a+b}{2}\right) \qquad (14.1)$$

To obtain a lower bound for $\int_a^b f(x)\,dx$ we again apply (1) to the right-hand member of (13), with $p(u) = (m-u)$ in the first integral and $u-m$ in the second. The result is

$$\int_a^b f(x)\,dx > (b-a)\frac{f(a)+4f(m)+f(b)}{6} + \frac{(b-a)^2}{144}[f'(b)-f'(a)] - \frac{(b-a)^3}{288}[f''(b)+f''(a)] \quad (14.2)$$

$$f'''(x) \text{ increasing in } (a,b) \qquad \left(m=\frac{a+b}{2}\right)$$

The signs of the inequalities (14.1) and (14.2) are reversed if $f'''(x)$ is decreasing in (a, b).

As an example, we take again $f(x) = x^4$, $a = 1$, $b = 2$, $I = \int_1^2 x^4\,dx = 6.20$. Formulas (14.1) and (14.2) give $6.1944 < I < 6.2032$. Kowalewski's formulas (l.c., p. 51) give here $6.1875 < I < 6.2084$, in this case under the same assumption, i.e. that $f'''(x)$ is monotonic in (a, b).

Formulas (12), (14.1) and (14.2) were derived under the assumptions that $f''(x)$ or $f'''(x)$ are monotonic in (a, b). However, this restrictive assumption is compensated for by two facts: first, our formulas give both the lower and the upper bound; secondly, the usual estimates of the error in the Trapezoidal and Simpson's Rules are one-sided (i.e. they give the upper bound only), and in addition they require the knowledge of the upper bound in (a, b) of $|f''(x)|$ or $|f^{(iv)}(x)|$.

Assume now that the interval (a, b) is divided into n equal subintervals of length h, the points of division being $a = a_0, a_1, \cdots, a_n = b$. Letting $a_{\nu+\frac{1}{2}} = \frac{a_\nu + a_{\nu+1}}{2}$, we obtain, from (14.1) and (14.2),

$$\int_a^b f(x)\,dx < \frac{h}{8}\sum_{\nu=0}^{n-1}[f(a_\nu)+6f(a_{\nu+\frac{1}{2}})+f(a_{\nu+1})] + \frac{h^3}{96}\sum_{\nu=0}^{n-1} f''(a_{\nu+\frac{1}{2}}), \quad (15.1)$$

$$f'''(x) \text{ increasing in } (a,b)$$

$$\int_a^b f(x)\,dx > \frac{h}{6}\sum_{\nu=0}^{n-1}[f(a_\nu)+4f(a_{\nu+\frac{1}{2}})+f(a_{\nu+1})] + \frac{h^2}{144}[f'(b)-f'(a)] - \frac{h^3}{288}\sum_{\nu=0}^{n-1}[f''(a_\nu)+f''(a_{\nu+1})] \quad (15.2)$$

$$f'''(x) \text{ increasing in } (a,b).$$

As a final numerical example, we take $f(x) = 1/(1+x)$, $a = 0$, $b = 1$, $n = 5$. Formulas (15.1) and (15.2) give $.6931451 < \log 2 < .6931485$. Kowalewski's formulas (l.c. p. 51) give $.6931428 < \log 2 < .6931503$. [$\log 2 = .69314718\cdots$].

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VORTICES IN FLUID FLOW

By

H. Poritsky

1. **Fundamentals of Hydrodynamics of Perfect Fluids.** The equations of hydrodynamics comprise:

(1) the equation of state, $p = p(\rho)$,

(2) the equation of continuity, $\partial\rho/\partial t + \nabla\cdot(\rho\mathbf{v}) = 0$,

(3) the equation of forces, $\rho\mathbf{a} = \mathbf{F}$.

The "equation of state," in the form shown in (1), is an assumed relation between the pressure p and the density ρ. The "equation of continuity" is really a misnomer and should be called the "equation of conservation of mass." The "equation of forces" is a statement of Newton's second law, $\mathbf{a}$ being the acceleration and $\mathbf{F}$ the total force per unit volume of the fluid.

If the fluid is incompressible so that ρ is constant:

(4) $$\rho = \rho_0,$$

then (2) reduces to the condition of vanishing velocity divergence:

(5) $$\nabla\cdot\mathbf{v} = \operatorname{div}\mathbf{v} = 0.$$

It is customary to apply the assumption of incompressibility even to fairly compressible fluids like gases. The justification of this lies in the fact that compressibility causes propagation of disturbances in the fluid with the velocity of sound. Therefore any tendency to change the fluid density will dilute itself rapidly over a large space except in cases where a volume change is forced on fluids which are enclosed, as for instance in the case of a gas in a cylinder with a moving piston, or in case of objects which move with velocities comparable with the velocity of sound, where compression waves cannot get speed away from the moving object fast enough.

The force $\mathbf{F}$ in (3) has two components. First there is the external or body force that acts throughout the body of the fluid, for instance the force of gravity; then there is the component due to the various stresses or interactions between the different parts of the fluid. For a *perfect fluid* it is assumed that across any surface in the fluid the latter forces act normally, that is at right angles to the surface They are then known as *pressure* forces and may be characterized by the scalar pressure p. This assumption amounts to neglecting tangential or shear stresses which, as known, do occur in actual fluids and are said to be due to the fluid "viscosity.' This assumption is also applied to the boundary surface of immersed objects where the fluid is supposed to experience no shear or drag and to exert no drag upon the immersed object.

While neglecting shearing forces and the effect of viscosity would appear to be a serious defect, it turns out that one can go quite a ways by neglecting them,

and that is what will be done throughout most of the following, showing what conclusions can be arrived at on that basis. Some corrections due to viscosity will be described presently.

One relation that can be arrived at on the above basis is "Bernoulli's equation." For an incompressible fluid, for steady flow, and in the absence of body forces, this takes the form

$$\frac{v^2}{2} + \frac{p}{\rho} = \text{Const.} \tag{6}$$

along any stream line, where v denotes the magnitude of $\mathbf{v}$. If there are external forces per unit mass which themselves are derivable from a force potential V, then (6) is replaced by

$$\frac{v^2}{2} + \frac{p}{\rho} + V = \text{Const.} \tag{7}$$

It is possible to modify the Bernoulli equation so that it applies to a compressible fluid too, by replacing the term p/ρ by a proper function of density, and also to extend it to certain flows that are not steady but where the flow pattern changes with time.

2. **Examples of Vortices.** Turning to vortices, first, what is a vortex? Familiar simple examples are abundant. Suppose that instead of gulping down your coffee in one or two gulps in the morning, you drink it leisurely, and stir the coffee with a spoon by immersing it half way in, moving it forward slowly, and raising it at the same time. When the spoon is out, you will find that you have started vortices moving in the coffee. You can recognize them by the fact that they show up as little dimples on the surface. It's very easy to spot them—they move on till they almost reach the cup, then they separate and move away from each other.

A more obvious example is offered in paddling a canoe. If you dip the paddle, move it back, and take it out at the end of the stroke, you can observe a strong vortex where the edge of the paddle had been. It is a U-shaped vortex, at first extending down as far as the paddle had been at its deepest. It often shows up as a silvery line filled with a big air bubble which may break up later, but it can always be recognized by the dimples or depressions over the surface of the water where it enters and leaves that surface. The rotation around each end of the U is in the opposite direction.

Other examples of vortices are offered by smoke rings, which some smokers are very adept at blowing. Another example on a much larger scale is offered by hurricanes and by cyclones and anticyclones of meteorology. Astronomers even tell us that sun spots are also connected with powerful vortex motion.

3. **Definition of Vorticity. Rotational and Irrotational Flow.** Now to tackle the subject of vortices more scientifically, remembering Lord Kelvin's dictum, it is necessary to set up a technical definition which can be used as a basis for measuring vortices.

It will be noticed that all the vortices described above are connected with circulating motion. To measure the circulation of the fluid around a closed curve C one defines it as the integral

$$K = \int_C v_t \, ds, \tag{8}$$

where K is the "circulation over C" and v_t is the tangential component of the velocity $\mathbf{v}$ along C, ds being the element of length of C (see Fig. 1).

To get down to point-by-point measurement of this circulation, one shrinks the curve C down to a point through a series of curves outlining a surface through that point. The ratio of the circulation K to the area of the enclosed surface is found to approach a limit which may be taken as the measure of the circulation per unit area or of the local *vorticity* in the tangent plane. This is represented

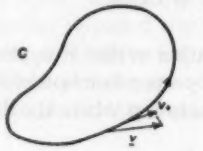

FIG. 1

FIG. 2

as a component of a vorticity vector $\boldsymbol{\omega}$ in the normal direction. Carrying out the mathematical steps outlined above, it is found that the vorticity is given by

$$\boldsymbol{\omega} = \nabla \times \mathbf{v} = \text{curl } \mathbf{v}. \tag{9}$$

One may regard $\boldsymbol{\omega}$ as a vector measure of the "rotativity" of the fluid, the component of $\boldsymbol{\omega}$ in any direction yielding the circulation per unit area normal to that direction.

A fluid motion for which

$$\boldsymbol{\omega} = \nabla \times \mathbf{v} = 0, \tag{10}$$

is known as "irrotational." On the other hand, if $\boldsymbol{\omega}$ does not vanish everywhere, the motion is "rotational" and possesses vorticity. From a theorem due to Lord Kelvin and Helmholtz, which will presently be stated, it follows that the fluid motion should often be of the irrotational type and that vorticity should never develop in fluids at all, or at least not in perfect fluids. Actually it is known that vortices do develop in fluid motion quite frequently. One of the main purposes of the following is to investigate this apparent paradox. After examining the theorem and finding that there is nothing really wrong with it, it will be necessary to do a certain amount of detective work to find out how the vortices manage to elude this theorem and slip in in spite of it.

4. **Properties of Irrotational Flow. The Laplace Equation.** Several interesting properties of irrotational flow will be pointed out.

First it is clear from the definition of ω that if the circulation K vanishes over any closed curve C, ω also vanishes identically and the flow is irrotational. The converse theorem, however, is not true. That is, the flow may be irrotational over a region R while the circulation K over some curves in R need not vanish. For $K = 0$ to follow from $\omega = 0$, the region R must be "simply connected," that is, such that any closed curve in it may be shrunk to a point without getting out of R. The inside of a sphere is an example of a simply-connected region and so is all of space. On the other hand, the inside of a torus is doubly connected, and so is the space from which the points of a closed curve C' have been omitted, since a curve C linking C' cannot be shrunk to a point without cutting C'.

A further property of irrotational flow over a simply-connected region R is the fact that the integral

$$\int_A^B v_t \, ds \tag{11}$$

(see Fig. 2) is *independent of path* of integration between A and B. The integrand is the same as in (8) but the integral now extends not over a closed curve but over a curve segment between A and B. Indeed, the difference of the integrals over two paths, both leading from A to B, is equal to the circulation over the closed curve made up of the two paths, and this circulation vanishes. If now the point A is kept fixed and the point B varied anywhere in space, then the value of this integral would be a function which depends only upon the position of B, and represents a true point function of B, say φ. This function φ is called the *velocity potential*, and the velocity can be derived from it by differentiation:

$$\mathbf{v} = \nabla\varphi = \operatorname{grad}\varphi. \tag{12}$$

Thus irrotational flow can also be described as flow which possesses a velocity potential.

For incompressible fluids substitution of (12) in (5) leads to

$$\nabla^2\varphi = 0. \tag{13}$$

Thus for incompressible irrotational flow the velocity potential φ is harmonic. Certain problems in fluid flow thus reduce themselves to the determination of solutions of the Laplace equation.

5. **Analogy Between Fields of Flow and Fields of Force.** The Laplace equation suggests immediately an analogy between fields of fluid flow and fields of force, since it is well known that it also applies to gravitational, electrostatic, and magnetostatic fields where it is satisfied by the proper force potential in regions free from matter, charge, and poles respectively. For fields of force the integral analogous to (8) and (11) (with $\mathbf{v}$ replaced by the force vector) now denotes the *work done* by the field in going around C or in going from A to B. For electric fields this work over a closed curve is known as the "electromotive force" and for a magnetic field the similar integral is called the "magnetomotive force." The analogy is especially helpful when applied between rotational

flow of fluids and magnetic fields produced by electric currents, in view of the fact that the calculation of the magnetic field due to a current is mathematically equivalent to the calculation of the velocity of a fluid induced by a vortex filament having the shape of the current.

One interesting property of vortex filaments is that they can never start or stop anywhere in space, and must therefore either form *closed curves* or else continue until they either hit the boundary of the liquid or go off to infinity. In this respect again they behave in a fashion similar to electric currents which

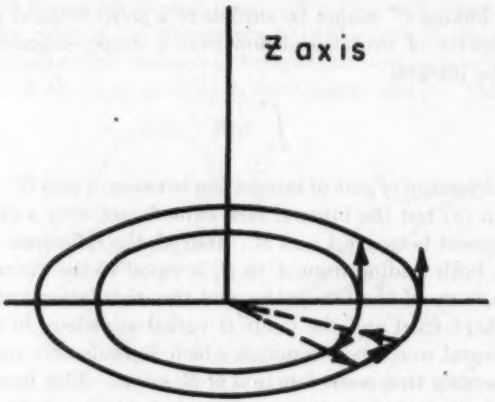

FIG. 3

are always closed or go off to infinity. For volume distributions of vortices this follows from the equation derived from (10):

$$\nabla \cdot \boldsymbol{\omega} = \nabla \cdot \nabla \times \mathbf{v} = 0. \tag{14}$$

As an example, the vortex filament started by a paddle extends over the water level around the edge of the paddle and back again to the water level.

An interesting application of this property of vortices will appear presently.

6. **Examples of Fluid Flow with Vortices.** Simple examples of vortices will now be given.

As the first example suppose that the particles of fluid describe circular paths having the z-axis as their axis, the magnitude of velocity varying *inversely* as the distance r from that axis (see Fig. 3): $v = C/r$, where C is a constant. It is easy to show by using closed paths indicated in Fig. 3 that ω vanishes for points not on the z-axis. Yet the circulation over a complete circle having the z-axis as axis is not zero but is equal to $K = 2\pi C$ no matter what the radius r is. This field is then irrotational everywhere not on the z-axis, yet it has a non-zero circulation for curves going around the z-axis. The explanation lies in the fact that the space outside the z-axis is doubly-connected. The flow may also be

regarded as due to a vortex filament with concentrated (infinite) vorticity along the z-axis. Recalling the analogy to magnetic fields it is seen that this flow field corresponds to the magnetic field produced by an electric current flowing along the z-axis.

The flow due to a vortex filament over a closed curve C' is described by the velocity potential $\varphi = C\Omega$ where Ω is the solid angle subtended by C'. Again the flow is irrotational everywhere not on C' but has a finite circulation $K = 4\pi C$, over curves linking C' once.

As a further example, the field given by $\mathbf{v} = Cy\mathbf{i}$, the velocity pointing in the x-direction and being proportional to y, while representing rectilinear flow, possesses a constant vorticity $\boldsymbol{\omega} = -C\mathbf{k}$ at all points.

7. **Kelvin's or Helmholtz's Circulation Theorem.** Assuming that the pressure p is a function of density (or that the density is constant), that the external forces possess a potential, and that the fluid transmits no shearing stresses, Lord Kelvin established the relation

$$\frac{dK}{dt} = 0; \tag{15}$$

here K is the circulation around a closed curve which moves with the fluid, every point of the curve following the path of a proper fluid particle, and d/dt denotes the rate of change of the circulation as the curve proceeds on its way. In other words, the circulation of the moving fluid can never be either increased or decreased and thus appears to be "grafted on" to the fluid. This endows vortices with a property of permanence: if a fluid has no circulation to begin with, it cannot acquire it; on the other hand, once it has somehow acquired circulation, it can never dispose of it.

So impressed were scientists by this circulation theorem that even atomic models were proposed consisting of vortices in a proper fluid, say in the ether. Presumably these vortices had been set up at creation and, guarded by the circulation theorem, retain their existence throughout the ages. With the various fluid theories of the ether nowadays banished by the empty spaces of Maxwell's and the quantum mechanics equations whose solutions serve for guiding photons and particles, the interest in such atomic models is at present only historic. (The shades of the past seem to be rising, however, in the electronic and nuclear spins!)

Turning to the examples in section 2 it will be noted that a vortex *could* be started in a cup of coffee by moving a spoon in it. Moreover, while the vortex, once started, *does* last a little while, yet if one keeps on looking for it long enough, eventually one can find no trace of it. Thus the coffee in the cup actually violates the circulation theorem flagrantly. One wonders why. Which one of the assumptions must be blamed?

8. **Applications of Kelvin's Theorem.** Let us examine some of the consequences of Kelvin's theorem. Consider as an example the superposition of two

vortices of the type shown in Fig. 3 with parallel vortex filaments. The velocity distribution due to the one vortex imparts a certain motion to the other one, and by Kelvin's theorem, as the particles along each vortex filament move, the circulation around each one does not change and must move with it. Thus while the vortices persist as vortices, they cannot stand still. If the circulations about each vortex are equal and in *opposite* directions, then, as shown in Fig. 4, both vortices move with a constant velocity in the same direction, the whole flow pattern undergoing a translation in that direction. This corresponds to the vortices which are started in paddling, or in stirring a cup of coffee with a spoon. On the other hand, if the circulation around each vortex is of the same magnitude and in the *same* direction, then as shown in Fig. 5, the velocities induced in the one vortex by the other one are in opposite directions, and the two vortices move at right angles to the line joining them, the whole figure executing a common rotation about the center of the line joining the two vortices.

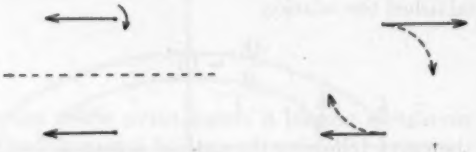

FIG. 4 FIG. 5

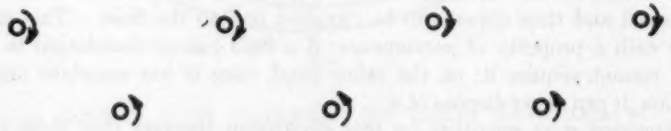

FIG. 6

Similarly other configurations of vortices in general execute complicated motions due to the velocities imparted to any one vortex by the other vortices. A feat for adept smokers is to blow one smoke ring through another one. Actually it can be shown that two circular vortices of the same strength and having a common axis will continually overtake each other and play a kind of leap-frog game. The one in the back catches up with the one in front of it and passes through it, then slows down only to be overtaken by the other ring, etc. This phenomenon essentially depends upon the velocity induced in the one ring by the other one.

9. **Stability and Instability of Certain Vortex Configurations.** Due to the mutual effect of vortices on each other, certain configurations of vortices are inherently *unstable*. Thus *periodic arrays* of parallel line vortices obtained by repeating the pattern of the vortices of Fig. 4 in the horizontal direction have been shown by von Karman to be *unstable* in the sense that a small initial

disturbance in the regular arrangement will grow and cause the vortices to scatter. On the other hand, it has been shown by von Karman that a periodic array such as shown in Fig. 6 is stable provided that a proper ratio exists between the horizontal and vertical spacings of the vortices. This array is known as a "Karman vortex street."

Again, a *surface distribution* of vortices is often *unstable*. Thus a distribution of vortex lines over a plane is unstable in the sense that if a small normal disturbance is started in the plane it will grow with time. Hence an incipient wave disturbance will grow, fold and roll itself up as shown in Fig. 7.

10. **Symmetric Irrotational Flow Around a Cylinder.** One of the main applications of Kelvin's theorem consists in reducing the motion of an incompressible fluid around an immersed object to a solution of Laplace's equation. Consider as an example a cylinder immersed in an incompressible fluid, both the cylinder and the fluid being initially at rest, and suppose that the cylinder is started at right angles to its axis and speeded up to a certain velocity. Since initially the fluid velocity is zero everywhere, the circulation K over any curve

FIG. 7. From Goldstein, *Modern Developments in Fluid Dynamics*, Vol. I, p. 30, Fig. 7

vanishes, and ω vanishes too. By Kelvin's circulation theorem then, the fluid cannot acquire any vorticity ever, and its velocity $\mathbf{v}$ can therefore be obtained at any time from a proper scalar potential φ by means of (12), this potential satisfying the Laplace equation (13). As a boundary condition, one generally assumes that the normal velocity of the fluid at the boundary is equal to the normal component of velocity of the moving cylinder. This amounts to supposing that the fluid gets out of the way of the moving cylinder in front and closes in behind without leaving any gaps. If the motion is described relative to axes fixed in the cylinder, then the fluid flow is tangential at the cylindrical boundary. Actually fluids are known to experience *no* relative motion at immersed bounding surfaces. It may be shown, however, that no solution of (13) satisfying this more stringent boundary condition is possible. On the other hand, the normal derivative condition determines a solution which is symmetric about the direction of motion and the axis at right angles to it and is shown in Fig. 8.

In this figure axes which are fixed relative to the cylinder have been used. It will be noticed that the flow is symmetrical about the vertical and horizontal axes through the center. The fluid parts in front of the cylinder at the stagna-

tion point A and reunites in back of it at C. The shape of the flow lines is independent of the velocity of the cylinder.

One disconcerting thought to be derived from examining Fig. 8 is that presumably one could find no shelter from a gale by hiding behind a tall cylindrical building, since there is apparently just as much wind velocity in the back as there is in the front! A further result which does not check experience is that there is no net reaction between the cylinder and the fluid. Comparing four points which are symmetrically situated about the two axes of symmetry of Fig. 8, it will be found that the velocity is the same at these four points. Therefore by Bernoulli's theorem, the pressure must be the same at these points. This means that the resultant of all the pressure forces at these four points would vanish and consequently, by integration, that the resultant of the total pressure forces on the cylinder as a whole would vanish. Apparently a perfect fluid would exert no reaction on an immersed object moving relative to it! This conclusion, which is of course at variance with known facts for actual fluids, is known as *d'Alembert's paradox*, and for a long time it served to put hydrodynamics in disrespect with practical engineers.

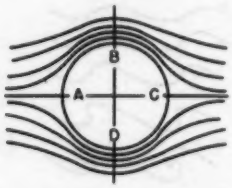

FIG. 8. From Goldstein, *Modern Developments in Fluid Dynamics*, Vol. I, p. 23, Fig. 3

The lack of reaction is no doubt due to neglecting viscous shear and allowing free slippage at the cylinder surface.

11. **Lift on a Cylinder.** It turns out that due to the multiple-connectivity of space around the cylinder, in addition to the type of flow shown in Fig. 8, there are also other possible types of irrotational ideal fluid flow around a cylinder. If one superposes upon the flow of Fig. 8 a vortex flow such as shown in Fig. 3 with the axis of the vortex along the axis of the cylinder, then both the boundary conditions and the proper differential equation are still completely satisfied (though the velocity potential is now multiple-valued). While the vorticity at each point vanishes, there is a non-zero circulation around any curve enclosing the cylinder. The resulting flow is shown in Fig. 9. The three cases shown in Fig. 9 correspond to various amounts of circulation, the circulation increasing from a to b to c.

It will be noted upon comparing Fig. 9 with Fig. 8 that the symmetry about a horizontal plane has disappeared and that the velocity is larger in magnitude on the upper side of the cylinder than it is on the lower side. Hence by Bernoulli's theorem the pressure will be lower on the upper side than on the lower.

As a result there will be a net force in the upward direction. This force is known as a *lift*. Calculation of its magnitude yields

$$L = K\rho v, \tag{16}$$

where K is the circulation, v the velocity of the cylinder relative to the fluid at infinity (or referring to Figs. 8 and 9, the velocity of the fluid at infinity relative to the stationary cylinder), and L the lift per unit axial length of the cylinder.

Thus it has been shown that even on the basis of ideal fluid flow a force may be exerted by the fluid upon an immersed object, namely a force of lift, this force depending upon the circulation of the fluid. While the theory is unable to show *how* such a circulation can originate in an initially quiescent fluid, but once the circulation *has been induced*, its effect shows up as a lift.

12. **Lift on an Airfoil.** Similar conclusions apply to cylindrical or prismatic objects of other than a circular cross section. Again it can be shown that irrotational ideal fluid flow around any cylinder with zero circulation around it

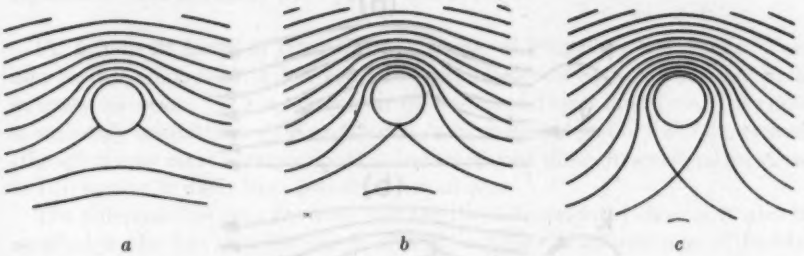

FIG. 9

will yield a zero resultant force on it, while if there is a non-zero circulation K about it will exert a lift on it still given by (16).

Applying these results to "wings" of airfoil section of infinite span, one finds, in particular, one type of flow which has no circulation and exerts no force on the airfoil. This type of flow is shown in Fig. 10a. There is another type of flow *with* circulation for which a lift will be experienced. How much lift there is will depend on the circulation K which is assumed.

A well known assumption in airfoil theory (due to Kutta and Joukovsky) is to suppose that the circulation K is such as to make the flow reunite smoothly at the tail end of the airfoil. Indeed, in Fig. 10a it will be seen that with $K = 0$ the flow reunites on the upper side of the airfoil and that therefore the fluid is actually supposed to flow back around the sharp edge. It may be shown that irrotational fluid flow must have infinite velocities when flowing around a sharp edge of this kind, as well as (by Bernoulli's theorem) negatively infinite pressures there. As the circulation superposed upon the fluid is increased, the two stagnation points at which the flow parts and reunites both move down until for a proper value of the circulation the point at which the flow reunites is at the

tail of the airfoil as shown in Fig. 10c. With more circulation yet, the point at which the flow reunites moves further from the edge on the downward side of the airfoil as shown in Fig. 10b and there is still infinite velocity at the sharp edge.

The above theory of lift is found to check experimental values fairly well for small "angles of attack," this angle being the angle between the largest chord and the direction of motion. The actual observed lifts are slightly smaller than the calculated ones, thus showing that the circulation is slightly less than predicted by the above assumption and that the flow reunites just above the tip. The above success of ideal fluid flow in predicting lifts has served more than anything else to bring theoretical hydrodynamics back to its own so that

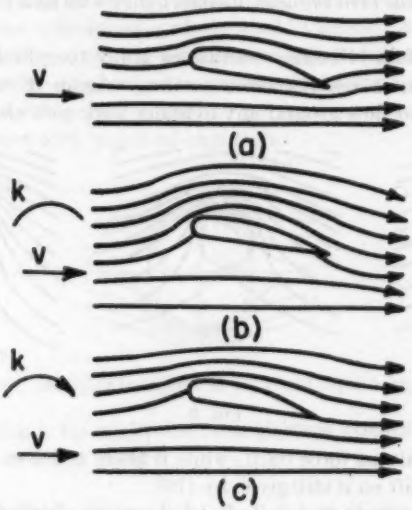

FIG. 10. From Goldstein, *Modern Developments in Fluid Dynamics*, Vol. I, p. 34, Fig. 10

engineers nowadays pay more attention to it than they used to. Especially in the design of airplanes is the theory of lift of various shapes utilized extensively. This theory has been well developed so as to enable the lift of various shapes of airfoils to be calculated. One method of calculation consists in mapping the outside of the airfoil section conformally on the outside of a circular cylinder. This mapping is effected by means of analytic functions of a complex variable which, as is known, play an important part in the two-dimensional solutions of the Laplace equation (13).

A somewhat more popular (and less correct) explanation for the lift may be given in the following way. For an airfoil it may be observed that the path which the fluid has to follow to go around the airfoil over the top is longer than the path below it. Therefore the fluid has to speed up while going over the

upper path and consequently with the greater velocity, there will result a lower pressure by Bernoulli's theorem.[1]

Examination of the pressure distribution shows that the lift is actually derived from a *decrease* of pressure or suction *above* the airfoil as well as from an *increase* of pressure *below* it. One can verify this by taking a small sheet of paper, holding one edge with one's finger tips near one's mouth, and blowing against it. One can lift it by suction created by blowing over the top as well as by pressure resulting by blowing under it.

As the angle of attack increases the measured lift increases too in agreement with the above theory until for points known as the "stalling angle" the lift suddenly decreases. The cause of this loss of lift will be examined presently.

Not only does the total lift for angles of attack below the stalling angle agree with the predicted value but the predicted distribution of pressure along the airfoil also is borne out by experiment, with the exception of the very large pressure values predicted by the theory at the trailing edge in case it is not infinitely sharp (see, for instance, Goldstein, *Modern Developments of Fluid Dynamics*, Oxford, (1938) Vol. I, p. 35 and p. 36.)

13. **Lift on an Airfoil of Finite Span. Thrust of Propellers and Fans.** Turn now to three-dimensional flow caused by the motion of objects which are finite in every direction. It may be shown that an irrotational fluid flow, if it closes in smoothly behind, can exert *no force* on an immersed object of finite dimensions (though it may exert a torque on it). In this respect three-dimensional solutions of (13) appear to differ from two-dimensional ones.

The difference between the two- and the three-dimensional cases may also be ascribed to the fact that the region around an airfoil of infinite span is doubly-connected, while the space around an object of finite dimensions is simply connected. Indeed, if one assumes that for an airfoil of finite but long span the flow near the middle is similar to the two-dimensional flow, and therefore possesses circulation, one is embarrassed by the fact that in a simply-connected space irrotational flow must possess vanishing circulation. Another way of stating the difficulty is by noting that for the two-dimensional case the vortex lines which the non-zero circulation implies, lie hidden inside the airfoil; with a finite span they can no longer do so. As stated previously, vortex lines must now either continue as closed curves or must go off to infinity.

A way of overcoming the above difficulties was discovered by Prandtl and Lanchester. They supposed that the vortex lines which go through the middle section of an airplane wing are shed off the tips and the sharp edge and trail behind it back to the point at which the motion first started. When the airplane first starts there is an initial vortex shed off the edge of the wings, and as the airplane proceeds the vortex lines stretch from its wings and get longer and longer, extending to and closing through that initial vortex. When the plane

[1] The fallacy in this argument is the assumption that the time of flow over the two paths must be the same.

comes to a stop a terminal vortex is shed. Fig. 11 shows schematically the aspect of the vortex lines as they trail behind the plane and close through the initial vortex.

As had been seen from previous simple examples, vortices in general do not stand still but each vortex imparts a proper motion to the others so that the whole vortex configuration tends to roll up. This is indicated schematically in Fig. 12 where 12a shows a cross section of the vortex filaments immediately behind the wing, while 12b shows a rolled up cross section farther down the stream.

The velocity induced by the trailing vortices has a further effect which does not appear in the two-dimensional case. There is a certain velocity component

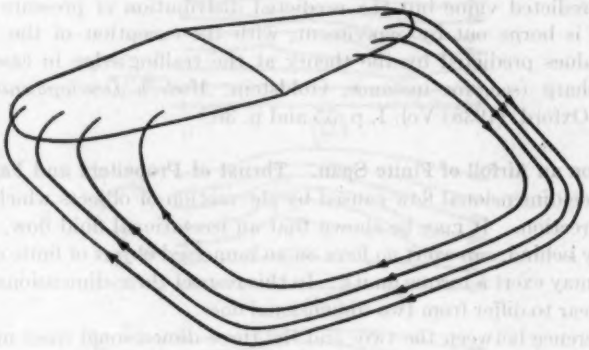

FIG. 11

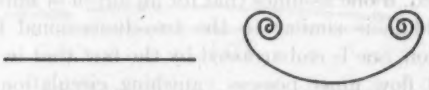

FIG. 12a FIG. 12b

induced by these vortices which results in an opposing force upon the airfoil. This result does not violate the previously stated conclusion regarding zero drag of an ideal fluid flow because in the latter case it had been assumed that the flow closes in smoothly behind the object, whereas in the present case, over the surface made up of trailing vortex filaments there is a certain discontinuity in velocity. The work done in overcoming this induced drag goes into setting up of vorticity in the fluid over which the plane has passed. A certain amount of energy is thus spent on, so to speak, "sowing" vorticity over the region covered, and this energy is derived from reactions over the airfoil and shows up as a drag on the airplane. This drag is known as the *induced drag* and, as stated, will be experienced even in ideal fluids.

The reaction of the moving airfoil on the fluid due to the lift component shows

up as a continually increasing downward fluid momentum; the reaction due to the induced drag, as a growing forward fluid momentum.

In addition to the induced drag there is a further skin friction drag due to viscosity. The division of the drag into these two parts is somewhat similar to the division of the resistance of a boat into *wave resistance*, energy being spent in developing a traveling wave pattern, and into *skin friction resistance*.

A fairly good demonstration of the rolling up of the vortex sheet behind an airplane may be observed in sky-writing, which is so often practiced nowadays in advertising. It will be noticed that the jet of smoke very rapidly enlarges and rolls up behind the plane until it reaches a finite width, thereafter remaining relatively stationary.

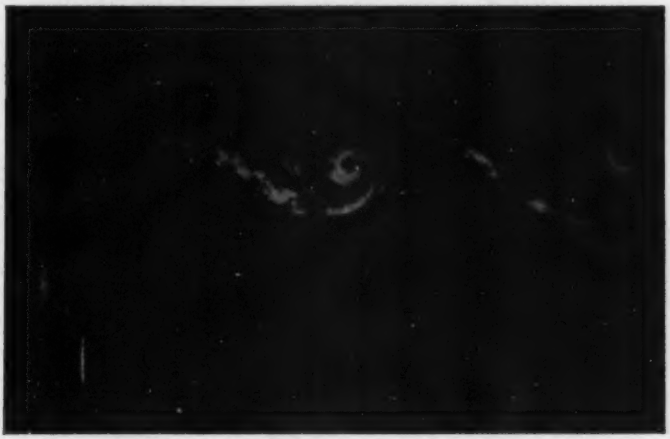

FIG. 13. From Valensi, "Etude des Champs Aerodynamiques par la Methode des Filets de Fumee," *Proceedings of the Fifth International Congress for Applied Mechanics*, edited by den Hartog and Peters, (Wiley, 1938), p. 521, Fig. 1

It is interesting to note that Kelvin's circulation theorem is in no way violated by the appearance of the trailing vortex sheet. It *is* true that initially, before the starting of the airfoil, the circulation around any closed curve vanishes, while after the motion has proceeded, closed curves with non-zero circulation may be found. However, any curve C which enclosed the airfoil initially, is deformed into a long curve ever enclosing the airfoil as well as the initial vortex and hence still has zero circulation. On the other hand, the closed curves which link the trailing vortex lines and show non-vanishing circulation, if traced back to the initial time, will be found to represent *open* curve segments.

Another way of stating the matter is as follows. In applying Kelvin's theorem it was tacitly assumed that closed curves remain closed as the motion of the fluid proceeds. However, the action of the finite airfoil may be compared to the cutting action of a knife. As the edges of the cut come together or "heal",

there is formed there a discontinuity or "scar". It is precisely along this discontinuity that the vortex lines manage to slip in and thus evade Kelvin's circulation theorem.

Turning to fans and propellers, it is found similarly that to account for the thrust imparted by them to the fluid, it is necessary to assume a definite non-zero circulation around the blades. The vortex lines which one thus postulates as

FIG. 14. From Edgerton and Killian, *Flash*, Hale, Cushman and Flint, (1939), p. 142

"bound vortex lines" inside the blades, continue into the fluid and trail behind as free vortices following down the slip stream. Fig. 13 shows a smoke picture of such vortices shed by a propeller of several blades. Fig. 14 gives a similar smoke picture of vortices shed by a fan. Fig. 15 shows the vortices behind a marine propeller. For a fan rotating in a space with little clearance, say for a centrifugal fan, the vortex lines may never show up as free vortex lines in the fluid proper, escaping directly from the blades to the housing.

It is interesting to point out that the exact determination of the shape of the

surface of discontinuity trailing from an airfoil as well as the distribution of vorticity along it, even for simple shapes of airfoils, never has been carried out except by highly approximate methods. The same statement applies to the surfaces of discontinuity trailing from fan and propeller blades. This is but one example of the many interesting and difficult unsolved problems of hydrodynamics.

14. **Comparison with Actual Flow Around a Cylinder.** Returning to the ideal flow around the cylinder which was shown in Fig. 8, let us compare it with photographs of actual flows. Fig. 16 which applies for low velocities shows that the predicted flow is not far from the flow actually observed. For larger veloci-

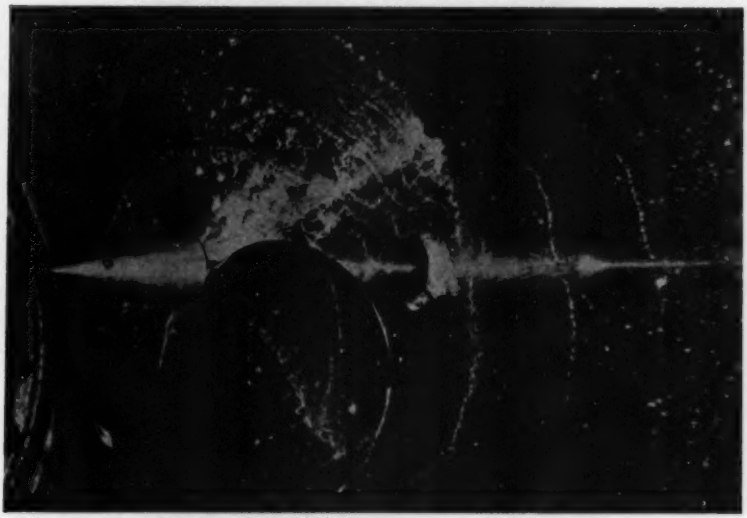

FIG. 15. From Edgerton and Killian, *Flash*, Hale, Cushman and Flint, (1939), p. 142

ties, however, it will be seen that the main flow departs from the cylindrical boundary in the back leaving a dead-water space As the velocity gets larger this dead-water space grows and vortices begin to appear in it which become more and more pronounced as the velocity increases. Eventually the vortices get to be as large as the cylinder itself and even larger. For higher speeds yet, this configuration for some reason becomes unstable and the vortices are shed alternately with new vortices growing out and being shed. The periodic pattern formed in the wake is shown in Fig. 17.

The explanation of some of these discrepancies from ideal flow may be given as follows.

Recall the ideal flow of Fig. 8. As pointed out there is a stagnation point at A, in front of the cylinder, and at C, immediately in the rear. A particle of

fluid which flows along the cylindrical surface starts with zero velocity at A, is speeded up down the pressure hill at the midpoint B between A and C, and then is slowed down again to zero velocity as it approaches C. Following Prandtl,

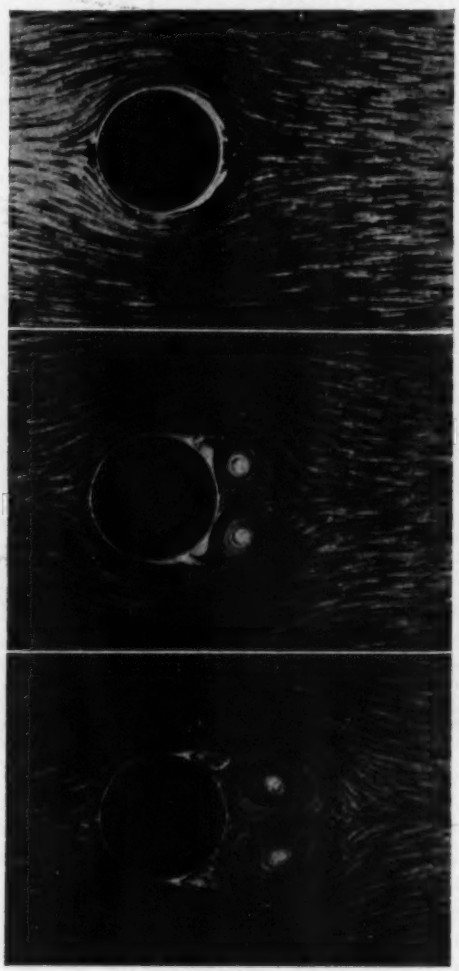

FIG. 16. From Prandtl and Tietjens, *Applied Hydro- and Aerodynamics*, McGraw Hill Book Company, (1934), p. 279

one may represent these facts by means of an analogy to a small ball which rolls down a hill as shown in Fig. 18, starting with a low velocity at A, the top of the hill, speeding up as it rolls down, and acquiring just enough speed to get up to

the top, C, again with practically zero velocity. If now the particle is slowed somehow while on its way, then it will not have quite enough energy to get up to the top, C, and will stop before it gets to it; it may then start rolling back. That is precisely what happens to the fluid particles flowing around a cylinder, the retardation being due to viscous drag exerted by the cylinder. As the fluid particles come to rest before reaching the second stagnation point, the main flow, not being able to wait for the laggard particles to make up their mind, so to speak, flows around them, thus departing from the cylindrical boundary. The dead-water spaces formed do not stay completely dead very long since the

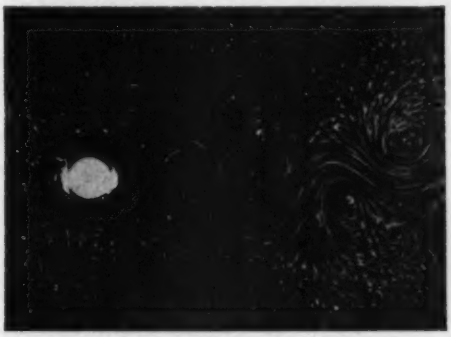

FIG. 17. Prandtl and Tietjens, *Applied Hydro- and Aerodynamics*, McGraw Hill Book Company, (1934), p. 303

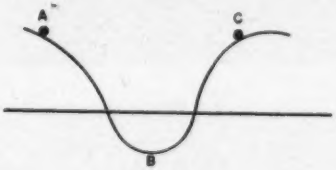

FIG. 18

viscous drag from the main flow gets a hold of their boundaries and induces the vortices which have been observed.

15. **Boundary Layer. Departure of Fluid from the Boundary in Regions of Divergent Flow. Stall of an Airfoil.** From the above it will be noticed that for low velocities and fluids of low viscosity the effect of the viscosity is felt near the surface of immersed solids only, in what is known as the "boundary layer". Immediately outside this layer the fluid appears to slide past the solid and to have the ideal irrotational aspect. This is especially true of the front end of the solid where the flow lines contract and the fluid flows down a pressure hill. On the "lee-side" where the fluid expands to resume its original cross section and

there is a pressure rise, there may be grave disturbances and departures from the ideal flow with the formation of dead-water spaces.

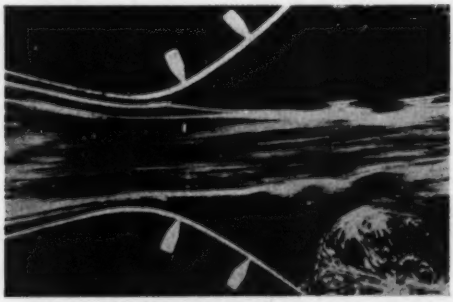

FIG. 19. From Prandtl and Tietjens, *Applied Hydro- and Aerodynamics*, McGraw Hill Book Company, (1934), p. 294

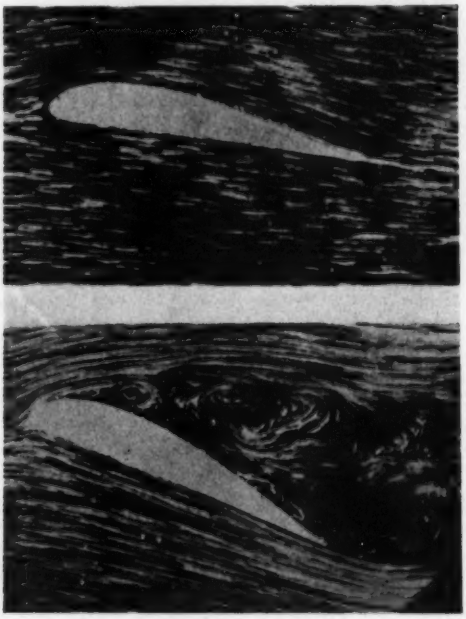

FIG. 20. From Goldstein, *Modern Developments in Fluid Dynamics*, Vol. I, Plate 12b, 12c

The difference between convergent and divergent flow is well shown in Fig. 19. The inability of the fluid to stay near the boundary in the divergent channel where there would be a boundary layer with a pressure rise is clearly shown here.

A "streamlined" shape refers to a shape which can be easily followed by a

fluid. Such a shape is offered by an airfoil section with an angle of attack below the angle of stall. As shown in Fig. 20a, the boundary condition that the fluid follows the surface (at least outside of their boundary layer) appears well substantiated in this case. Similarly the assumption of smooth flow off the tail end is well borne out. On the other hand, it appears from Fig. 20b that for an angle of attack larger than the stalling angle the fluid is unable to negotiate the boundary layer in the region of rising pressure and departs from the upper surface of the airfoil with the formation of stagnant spaces and eddies.

A phenomenon similar to the stall of an airfoil may occur for a propeller. In water, the failure to follow the blade contour is even more serious since it may result in cavitation—the formation of cavities filled with water vapor and dissolved air. This is shown in Fig. 15 where the cavities are clearly seen. After collapsing they continue as vortex filaments.

For bluff objects, for instance for a plate set at right angles to the flow, the tendency of the fluid to leave the boundary in the back is even more pronounced than for a cylinder.

16. **Ideal Fluid Flow with a Wake.** Ideal fluid flows which more nearly correspond with facts and provide for wakes and stagnant spaces, may be obtained. If the wake is of finite size and the flow reunites back of it, there is still no drag coefficient. If the dead-water space extends to infinity there results a drag which for a plate at right angles to the flow agrees not too badly with measured drag, and flow photographs show fair agreement even for the flow pattern near the plate. At the boundary of the wake of dead-water space the velocity is discontinuous and it must be regarded as a vortex sheet. Such vortex sheets as shown in Fig. 7 tend to roll up into periodic structure. It may be that this is the reason for the appearance of vortices along the wake boundary, as shown in Fig. 21.

The tendency for a sheet of discontinuity to form waves and wrap itself is of interest in meteorology, where two layers of air of different velocity and temperature come in contact with each other. It is responsible for certain periodic cloud formation. It also forms the basis of Bjerknes' theory of "depression" formed at the boundary between say a tropical mass of air moving east and a polar mass of air moving west. If a slight intrusion of the warm into the cold air mass starts at one point, it grows and is finally wrapped around by the cold air. The tendency for discontinuous flow to wrap itself may be noted by observing smoke as it rises from a cigarette resting on an ash tray. The smoke rises straight at first, then breaks out into waves and curls up.

17. **Vortices Shed by a Cylinder.** The periodic vortices behind cylindrical or other objects which have been shown in Fig. 17 essentially form a Karman vortex street (see Fig. 6). Von Karman showed that resistance coefficients based upon this shedding of vortices again agrees with experiment fairly well.

At the time when the vortices are shed in the cylinder, the latter experiences a certain kick in the transverse direction. Since the circulation around the

curve enclosing the cylinder then suddenly changes, there will be a tendency for the cylinder to oscillate at right angles to the motion. If you pass a rod through water rapidly you can feel this transverse oscillation tendency. It is due to the shedding of vortices and may even be observed while taking a bath if you immerse one finger and move it through the water. This phenomenon occurs in case of wires, say telephone or telegraph wires, where it is responsible for their humming when the wind is blowing. The tones originated by the shedding of these vortices are known as eolian tones. Their pitch is determined by the frequency of shedding which depends upon the speed of the wind, and increases with wind velocity.

18. **Does Viscosity Generate Vorticity?** To what extent is viscosity responsible for the appearance of vortices? Some light on this subject is shed by the

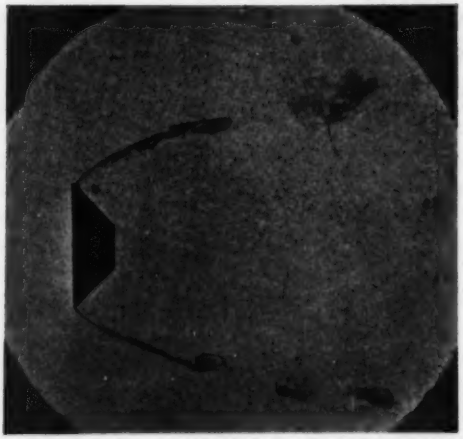

FIG. 21. From Goldstein, *Modern Developments in Fluid Dynamics*, Vol. II, Plate 33c

following example. Suppose that a rectilinear vortex filament (of the type shown in Fig. 3) is started in a *viscous* fluid. It may be shown that the effect of the viscosity consists in causing the vorticity to diffuse from its initial concentration along the z-axis, the *spreading* or diffusion of vorticity occurring in a manner similar to the conduction of heat from a concentrated instantaneous heat source, but without causing any increase or decrease in its total amount. Yet, oppositely rotating vortices may destroy each other as they diffuse due to viscosity. Again, viscosity combined with the motion of solid boundaries may cause the appearance of vortices. Consider as an example a pail, half filled with water, initially stationary. If the pail is started rotating about its axis and kept at a constant speed, the fluid will eventually be found to rotate as a rigid body, and thus acquire vorticity. Since an actual fluid actually never slips past an immersed surface, the layer right next to the pail surface acquires circulation, which by means of viscosity diffuses into the rest of the fluid.

19. Can Vortices Arise Due to the Failure of a Fluid to Satisfy the Assumed Curve in the Equation-of-State Plane? We next turn to the assumed equation of state, eq. (1). Can its assumption be responsible for the failure of actual fluids to obey Kelvin's theorem? There can be no question that if the representative point were not confined to a proper curve in the (p, ρ)-plane, vorticity may appear in the fluid. All kinds of cases of natural convection point to that conclusion, where due to larger buoyancy due to increased temperature, even the gravity forces which possess potentials may start circulations. Such are the convection streams observed near a hot object, and a great deal of atmospheric circulation is due to it. A simple example can be witnessed in opening a door between two rooms in which the air is stationary but at different temperatures. The moment the door is opened the circulation will start between the rooms, the warm air flowing into the cold room through the upper half of the door and the cold air returning through the lower half. The apparent violation of Kelvin's theorem is merely due to the fact that the air is not all along the same curve of the (p, ρ)-plane.

20. Summary. In summarizing the above, we saw that while Kelvin's Circulation Theorem, so to speak, chases out vortices from many types of fluid flow, actually vortices slip in in many ways. In case of a finite airfoil the manner of their re-entering is by means of the surface of discontinuity which trails back of the airfoil, namely the surface along which the flow reunites after it had been separated by the airfoil. For airfoils above the stalling angle and behind cylinders and bluff objects, the flow departs from the solid surfaces and the resulting discontinuities in flow are equivalent to vortex distributions. Viscosity is not by itself a cause of vorticity but it causes vorticity to diffuse throughout the fluid; however, combined with the effect of immersed solids it may cause vortices to appear. Failure of the fluid to stay along a single curve in the equation-of-state plane may cause vortices to appear. Vortices are responsible for lift of an airfoil and for the thrust of fans and propellers.

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LINEAR DIFFERENTIAL EQUATIONS WITH TWO-TERM RECURRENCE FORMULAS[1]

By Henry Scheffé

1. Introduction. In the nomenclature of the theory of linear differential equations (DE's), a *recurrence formula* may mean a formula relating either (a) solutions corresponding to different values of a parameter, or (b) the coefficients in a formal power series solution. We shall use the term exclusively in the sense (b) but postpone its precise definition to §2. Many of the DE's of applied mathematics have a two-term recurrence formula (TTRF), for example, the equations of Gauss, Legendre, Bessel, and Hermite.

Necessary and sufficient conditions on the coefficients of the DE for the existence of a TTRF are established under a suitable hypothesis in the small in §2. The addition of another hypothesis permits the determination of the type of the DE in the large in §3. In §4 simple transformations are exhibited which carry this type of DE into the generalized hypergeometric equation. The importance of this result lies in the fact that the latter equation has become the subject of an extensive literature[2], much of it devoted to the asymptotic behavior of the solutions.

The recurrence formula of a DE is defined relative to a chosen point in the plane of the independent variable. The question whether a DE can have a TTRF relative to several distinct finite points is considered in §5. The paper concludes with some specializations to second order DE's in §6. A discussion of a certain difficulty connected with the definition of the recurrence formula has been added in the form of an appendix.

2. Study in the small. We consider DE's

$$\sum_{j=0}^{n} p_j(z) w^{(j)} = 0 \tag{1}$$

satisfying

Hypothesis 1. *The coefficients $p_j(z)$ of the DE* (1) *are analytic in a sufficiently small neighborhood of some point z_0 of the extended z-plane, except possibly for poles at z_0, and are not all identically zero in this neighborhood.*

This permits of the presence of an ordinary point of the DE at z_0, or of a regular or irregular singularity.[3] Following Frobenius[4] we substitute in the DE a series

$$w = \sum_{r=0}^{\infty} c_r \xi^{r+\tau},$$

[1] Presented to the American Mathematical Society, April 11, 1941.

[2] See footnote 10. Other articles after 1907 are listed as follows in *Jahrbuch über die Fortschritte der Mathematik*: vols. 39–44, part 7, ch. 2 A; vol. 45, part 7, ch. 3 A; after vol. 45, part 4, ch. 6 B.

[3] These terms are defined just before Corollary 4 below.

[4] *Ueber die Integration der linearen Differentialgleichungen durch Reihen*, Journal für die reine und angewandte Mathematik, vol. 76 (1873), pp. 214–235.

where

$$\xi \equiv (z - a)^{\delta},$$

$$\delta \equiv \begin{cases} 1 \text{ if } z_0 \neq \infty, \\ -1 \text{ if } z_0 = \infty, \end{cases}$$

and $a = z_0$ if $z_0 \neq \infty$, but a is left unspecified at present if $z_0 = \infty$. The substitution is purely formal: neither now nor later do we assume that the series converges, nor even that the coefficients c_r can be successively determined. The operations encountered in the formal substitution are defined to be those which would be valid if the series converged. Then

$$w^{(j)} = \sum_{r=0}^{\infty} [\delta(\nu + r)]_j c_r \xi^{\nu+r-\delta j}, \tag{2}$$

where we use the Pochhammer notation for the factorial polynomials

$$[x]_j \equiv x(x - 1) \cdots (x - j + 1), \qquad j = 1, 2, 3, \cdots,$$

$$[x]_0 \equiv 1.$$

Before substituting the series we first multiply the DE (1) through by a suitable integral power of ξ, say ξ^M, so that it becomes

$$\sum_{j=0}^{n} q_j(z) \xi^{\delta j} w^{(j)} = 0, \tag{3}$$

where

$$q_j(z) = \sum_{k=0}^{\infty} q_{jk} \xi^k, \tag{4}$$

and

$$q_{j0} \neq 0 \text{ for some } j. \tag{5}$$

The convergence of the series (4) in some neighborhood of z_0 and the possibility of satisfying condition (5) by proper choice of M are consequences of Hypothesis 1.

DEFINITION 1. *The formula obtained by equating to zero the coefficient of* $\xi^{\mu+\nu}$, $\mu = 0, 1, 2, \cdots$, *when the series* (2), (4) *are substituted in* (3), *is called the recurrence formula of the DE*(1) *relative to the point* a *if* $z_0 \neq \infty$, *relative to the point* ∞_a *if* $z_0 = \infty$.

The notation ∞_a in the latter case is necessitated by the dependence of the recurrence formula (unlike the indicial equation) on the value of a chosen. In either case the recurrence formula is found to be

$$\sum_{l=0}^{n} Q_l(\delta[\mu - l + r]) c_{\mu-l} = 0, \tag{6}$$

where

$$Q_l(x) \equiv \sum_{j=0}^{n} q_{jl}[x]_j. \tag{7}$$

We note that because of (5),

$$Q_0(x) \not\equiv 0. \tag{8}$$

DEFINITION 2. *The DE* (1) *is said to have a TTRF relative to a if* $z_0 \neq \infty$, *to* ∞_a *if* $z_0 = \infty$, *if the recurrence formula is of the type*

$$f(\nu, r)c_\nu + g(\nu, r)c_{\nu-h} = 0$$

for $\nu = h, h+1, h+2, \cdots$, *where* h *is some positive integer, and neither* $f(\nu, r)$ *nor* $g(\nu, r)$ *is identically zero in both* ν *and* r.

Because of (8) we see that necessary and sufficient conditions that (6) be a TTRF are that

$$Q_h(x) \not\equiv 0, \tag{9}$$

$$Q_l(x) = 0, \qquad l \neq 0, h. \tag{10}$$

The necessity of (9) is obvious, that of (10) may be argued as follows: The $Q_l(x)$ are polynomials of degree $\leqq n$ and can vanish for $x = \delta(\mu - l + r)$, $\mu - l = h, h+1, \cdots, h+n$, and fixed r, only if $Q_l(x) \equiv 0$. (9) and (10) are equivalent to

$$q_{jh} \not\equiv 0 \quad \text{for some } j,$$

$$q_{jl} = 0, \qquad j = 0, 1, \cdots, n, \qquad l \neq 0, h.$$

The coefficients $p_j(z)$ are then of the form

$$p_j(z) = (q_{j0} + q_{jh}\xi^h)\xi^{hj-M}.$$

If $z_0 \neq \infty$,

$$p_j(z) = [q_{j0} + q_{jh}(z-a)^h](z-a)^{j-M},$$

but if $z_0 = \infty$,

$$p_j(z) = [q_{jh} + q_{j0}(z-a)^h](z-a)^{j+M-h}.$$

Hence in either case we have

THEOREM 1. *Necessary*[5] *and sufficient conditions that a DE satisfying Hypothesis* 1 *have a TTRF relative to the point* $a = z_0$ *if* $z_0 \neq \infty$, *to the point* ∞_a *if* $z_0 = \infty$, *are that in some neighborhood of* z_0

$$p_j(z) = [S_j - T_j(z-a)^h](z-a)^{j-m}, \tag{11}$$

where m *and* $h > 0$ *are integers, and* $S_i T_j \not\equiv 0$ *for some* i, j.

3. Extension to the large. Hypothesis 1 obviously allows us to assert (11) only in some neighborhood of z_0. We now add

[5] See Appendix.

HYPOTHESIS 2. *For all z the coefficients $p_j(z)$ of the DE* (1) *coincide with the monogenic analytic functions generated by analytic continuation of $p_j(z)$ from the neighborhood of z_0.*

Since (11) may then be asserted for all z, we have

COROLLARY 1. *If a DE satisfies Hypotheses* 1 *and* 2 *with* $\begin{Bmatrix} z_0 \neq \infty \\ z_0 = \infty \end{Bmatrix}$ *and has a TTRF relative to the point* $\begin{Bmatrix} a = z_0 \\ \infty_a \end{Bmatrix}$, *it also has a TTRF relative to the point* $\begin{Bmatrix} \infty_a \\ a \end{Bmatrix}$.

It follows that in stipulating a TTRF for DE's satisfying Hypotheses 1 and 2, no generality is lost in framing

HYPOTHESIS 3. *The DE* (1) *has a TTRF relative to the point $a = z_0 \neq \infty$.*

Since the rest of this paper deals with equations satisfying Hypotheses 1, 2, 3, it will be convenient to introduce the terminology of

DEFINITION 3. *A DE satisfying Hypotheses* 1, 2, 3, *will be called an M-equation.*[6]

In Theorem 1 let s be the maximum j for which $S_j \neq 0$, t be the maximum j for which $T_j \neq 0$. The order n of the equation is then the larger of s, t, or else $n = s = t$.

COROLLARY 2. *A necessary and sufficient condition that a DE be an M-equation is that it be of the type*[7]

$$\xi^{-m}\left\{\sum_{j=0}^{s} S_j \xi^j w^{(j)} - \xi^h \sum_{j=0}^{t} T_j \xi^j w^{(j)}\right\} = 0, \tag{12}$$

where $S_s T_t \neq 0$, m and $h > 0$ are integers, and $\xi \equiv z - a$.

If we define[8] the polynomials

$$\begin{aligned} S(x) &\equiv \sum_{j=0}^{s} S_j [x]_j = S_s \prod_{j=1}^{s} (x - \sigma_j), \\ T(x) &\equiv \sum_{j=0}^{t} T_j [x]_j = T_t \prod_{j=1}^{t} (x + \tau_j), \end{aligned} \tag{13}$$

[6] The equation (12) below was studied by R. Most, *Ueber eine lineare Differentialgleichung m-ter Ordnung*, Zeitschrift für Mathematik und Physik, vol. 15 (1870), pp. 427–450. He observed that in certain cases the solutions could be expressed as generalized hypergeometric functions.

[7] The special case of (12) when $s = t$ was derived under more severe hypotheses by R. Weth, *Ueber eine Verallgemeinerung der Gauss' schen Differentialgleichung*, Bericht der Realschule zu Basel, Basel, 1899. The form (our $s = 2$, $t = 1$) of the second order DE with a TTRF and with certain restrictions on the singularities was given without proof by W. Jacobsthal, *Ueber die asymptotische Darstellung von Lösungen lineare Differentialgleichungen*, Mathematische Annalen, vol. 56 (1903), pp. 129–154.

[8] The symbol $\prod_{j=1}^{0}$ here and elsewhere is defined to signify unity.

the recurrence formula (6) becomes, for $\nu \geqq h$,

$$S(\nu + r)c_\nu - T(\nu - h + r)c_{\nu-h} = 0 \tag{14}$$

relative to the point a, and

$$-T(-\nu - r)c_\nu + S(-\nu + h - r)c_{\nu-h} = 0 \tag{15}$$

relative to the point ∞_a. Since s and t are the actual degrees of the polynomials $S(x)$, $T(x)$, Cauchy's ratio test together with (14), (15) leads to

COROLLARY 3. *For any non-terminating series*

$$\sum_{k=0}^{\infty} c_{kh}(z - a)^{kh+r} \tag{16}$$

or

$$\sum_{k=0}^{\infty} c_{kh}(z - a)^{-kh-r} \tag{17}$$

formally satisfying an M-equation, the circle of convergence is the following,[9] *where* $R \equiv |S_s/T_t|^{1/h}$:

Case	Circle for (16)	Circle for (17)
$s > t$	$\|z\| < \infty$	$z = \infty$
$s < t$	$z = a$	$\|z - a\| > 0$
$s = t$	$\|z - a\| < R$	$\|z - a\| > R$

The singularities of an M-equation may now be classified. Suppose a DE is written in the form (1) with $p_n(z) \equiv 1$, and that the $p_j(z)$ are analytic at $a \neq \infty$ or have poles at a. Then a is called an *ordinary point* of the DE if the $p_j(z)$ are all analytic there, otherwise it is called a *singularity* of the DE. A singularity is called *regular* if $p_j(z) \equiv (z - a)^{j-n}\phi_j(z)$ with $\phi_j(z)$ analytic at a, *irregular* otherwise. The exponents at a are defined to be the roots of the polynomial $Q_0(x)$ defined by (7). The classification of the point ∞_a as an ordinary point, regular or irregular singularity and the exponents at ∞_a are defined to be those at $\zeta = 0$ for the DE obtained by transforming $\zeta = (z - a)^{-1}$. Since these particular characteristics are then invariant with respect to a, the subscript may be dropped from ∞_a.

COROLLARY 4. *The only possible singularities of an M-equation are at the points given in the following table. Classification I denotes that the point may be either a regular singularity or an ordinary point; II, that it is always an irregular singularity.*

[9] The behavior of the series on the circle of convergence in the case $s = t$ is treated by R. Weth, loc. cit.

Case	Point	Classification	Exponents
$s > t$	a	I	$\sigma_1, \sigma_2, \cdots, \sigma_s$
	∞	II	$\tau_1, \tau_2, \cdots, \tau_t$
$s < t$	a	II	$\sigma_1, \sigma_2, \cdots, \sigma_s$
	∞	I	$\tau_1, \tau_2, \cdots, \tau_t$
$s = t$	a	I	$\sigma_1, \sigma_2, \cdots, \sigma_s$
	$a + \omega_i$, $i = 1, 2, \cdots, h$, where ω_i are roots of $\omega^h = S_s/T_t$	I	$0, 1, \cdots, s-2,$ $s - 1 + \left(\frac{S_{s-1}}{S_s} - \frac{T_{t-1}}{T_t}\right)/h$
	∞	I	$\tau_1, \tau_2, \cdots, \tau_t$

A DE (1) all of whose singularities are regular is called a Riemann P-equation.

COROLLARY 5. *If a Riemann P-equation has a TTRF relative to some point, its singularities and the corresponding exponents must be distributed as shown in the above table for $s = t$.*

4. **Transformation to the generalized hypergeometric equation.** We take the generalized hypergeometric equation (GHE) in the form[8] given by Barnes,[10] where $D \equiv \zeta d/d\zeta$:

$$\left\{\prod_{j=1}^{p}(D + \alpha_j) - \frac{d}{d\zeta}\prod_{j=1}^{q}(D + \rho_j - 1)\right\}\psi = 0 \tag{18}$$

We shall call p and q the *indices* and α_j, ρ_j the *parameters*. The only possible singularities of this equation in the finite ζ-plane are at $\zeta = 0$ and 1. We shall succeed in transforming the M-equation into (18) with $p \leqq q + 1$. In this case the points 0, 1 are regular singularities or ordinary points.[11] We shall have occasion to refer to *Gauss' hypergeometric equation*; this is the special case of (18) with $p = 2, q = 1$:

$$(\zeta^2 - \zeta)\frac{d^2\psi}{d\zeta^2} + [(\alpha_1 + \alpha_2 + 1)\zeta - \rho_1]\frac{d\psi}{d\zeta} + \alpha_1\alpha_2\psi = 0. \tag{19}$$

[10] *The asymptotic expansions of integral functions defined by generalized hypergeometric series*, Proceedings of the London Mathematical Society, (2), vol. 5 (1907), pp. 59–116.

[11] Since the GHE is itself a special case of an M-equation, Corollary 4 may be applied to classify its singularities. In the notation of Corollary 4, $\zeta = 0$ is (I) if $p \leqq q + 1$, $\zeta = 1$ is (I) if $p = q + 1$ and definitely an ordinary point if $p < q + 1$.

We now consider transformations of the type

$$\zeta = \beta\xi^{\kappa}, \qquad w = \xi^{\gamma}\psi, \tag{20}$$

where $\xi \equiv z - a$. Then

$$\xi w' = \xi^{\gamma}[\kappa D + \gamma]\psi,$$

and it is easily proved by induction that

$$\xi^{j} w^{(j)} = \xi^{\gamma}[\kappa D + \gamma]_{j}\psi.$$

The equation (12) becomes, after dividing out $\xi^{\gamma-m}$,

$$\{S(\kappa D + \gamma) - \xi^{h}T(\kappa D + \gamma)\}\psi = 0,$$

where the polynomials $S(x)$, $T(x)$ are defined by (13). Employing the factored form of the polynomials, we have

$$\left\{S_s \prod_{j=1}^{s} (\kappa D + \gamma - \sigma_j) - \xi^{h} T_t \prod_{j=1}^{t} (\kappa D + \gamma + \tau_j)\right\}\psi = 0,$$

or

$$\left\{S_s \kappa^{s} \prod_{j=1}^{s} \left(D + \frac{\gamma - \sigma_j}{\kappa}\right) - T_t \kappa^{t} \xi^{h} \prod_{j=1}^{t} \left(D + \frac{\gamma + \tau_j}{\kappa}\right)\right\}\psi = 0. \tag{21}$$

Two cases arise; if $s = t$ either transformation may be used.

Case 1. $s \geqq t$. We specialize[12] the transformation (20) with

$$\beta = T_t h^{t-s}/S_s, \qquad \kappa = h, \qquad \gamma = \sigma_s. \tag{22}$$

The equation (21) then takes the form

$$\left\{\prod_{j=1}^{s} \left(D + \frac{\sigma_s - \sigma_j}{h}\right) - \zeta \prod_{j=1}^{t} \left(D + \frac{\sigma_s + \tau_t}{h}\right)\right\}\psi = 0.$$

The last factor in the first product is simply $D = \zeta d/d\zeta$, and since the factors may be permuted, we can write this first. If we then divide out $-\zeta$, the result is (18) with

$$\begin{aligned} &p = t, \qquad q = s - 1, \\ &\alpha_j = (\sigma_s + \tau_j)/h, \qquad j = 1, 2, \cdots, t, \\ &\rho_j = 1 + (\sigma_s - \sigma_j)/h, \qquad j = 1, 2, \cdots, s - 1. \end{aligned} \tag{23}$$

Case 2. $s \leqq t$. This time we take as the constants of the transformation

$$\beta = S_s(-h)^{s-t}/T_t, \qquad \kappa = -h, \qquad \gamma = -\tau_t. \tag{24}$$

[12] We take $\gamma = \sigma_s$ to simplify the notation. Since any root of $S(x)$ may be called σ_s, the number of distinct transformations given by our formulas in this case equals the number of distinct roots of $S(x)$; similarly, for the other case and $T(x)$.

A calculation similar to the preceding shows that (12) transforms into (18) with

$$p = s, \quad q = t - 1$$

$$\alpha_j = (\tau_t + \sigma_j)/h, \quad j = 1, 2, \cdots, s, \tag{25}$$

$$\rho_j = 1 + (\tau_t - \tau_j)/h, \quad j = 1, 2, \cdots, t - 1.$$

We summarize these results in

THEOREM 2. *An M-equation* (12) *can always be transformed into the GHE* (18) *with* $p \leqq q + 1$, *under a transformation* (20). *If* $\left\{\begin{matrix} s \geqq t \\ s \leqq t \end{matrix}\right\}$ *the constants of the transformation are given by* $\left\{\begin{matrix} (22) \\ (24) \end{matrix}\right\}$ *and the indices and parameters of the GHE by* $\left\{\begin{matrix} (23) \\ (25) \end{matrix}\right\}$.

5. **Differential equations with two-term recurrence formulas relative to several distinct finite points.** Gauss' equation (19) has a TTRF relative to two distinct finite points, namely $\zeta = 0$ and 1. We raise the question,[13] under what conditions can an M-equation have a TTRF relative to the finite point $\hat{a} \neq a$ as well as relative to a? From Theorem 1, necessary and sufficient conditions are that

$$p_i(z) \equiv [S_i - T_i\xi^h]\xi^{i-m} \equiv [\hat{S}_i - \hat{T}_i\hat{\xi}^{\hat{h}}]\hat{\xi}^{i-\hat{m}}$$

identically in z, where $\xi \equiv z - a$, $\hat{\xi} \equiv z - \hat{a}$; $h > 0$, m, $\hat{h} > 0$, $\hat{m}$ are integers; and $S_iT_j\hat{S}_k\hat{T}_l \neq 0$ for some i, j, k, l. We first note that the right member can have a pole only at $\hat{a}$ in the finite z-plane, the left member only at a, hence neither has a pole for $z \neq \infty$, and $S_j = 0$ for $j < m$, $T_j = 0$ for $j < m - h$. Next we note that if for any j, $|S_j| + |T_j| \neq 0$, the right member can have at most a simple zero at $z = a$. To prove $S_j = T_j = 0$ for $j > m + 1$, suppose the contrary, $|S_j| + |T_j| \neq 0$; then the middle member has a zero of order > 1 at $z = a$. Likewise we prove that if $h > 2$, so that j can take on values $m - h + 1 < j < m$, then for such j, $T_j = 0$. We have now found that the only possible non-zero coefficients in the middle member are S_{m+1}, S_m, T_{m+1}, T_m, T_{m-h+1}, T_{m-h}, and of these not both S nor all T can be zero. By further reasoning about the zeros, somewhat tedious because of the many special cases arising from the vanishing of coefficients, we may establish

COROLLARY 5. *A necessary and sufficient condition that an M-equation have a TTRF relative to two distinct finite points* a *and* $\hat{a}$ *is that* (*except for a constant factor not zero*) *it be one of the five following types with* $\xi \equiv z - a$, $\hat{\xi} \equiv z - \hat{a}$:

$$\text{(i)} \qquad \xi\hat{\xi}w^{(n)} + (A\xi + \hat{A}\hat{\xi})w^{(n-1)} + Bw^{(n-2)} = 0, \quad |A\hat{A}| + |B| \neq 0,$$

$$\text{(ii)} \qquad (2\hat{\xi} - \xi)\xi w^{(n)} + A\hat{\xi}w^{(n-1)} + Bw^{(n-2)} = 0, \quad |A| + |B| \neq 0,$$

[13] Incorrectly answered "never" by R. Weth, loc. cit.

(iii) $$(2\xi - \hat{\xi})\hat{\xi}w^{(n)} + A\xi w^{(n-1)} + Bw^{(n-2)} = 0, \qquad |A| + |B| \neq 0,$$

(iv) $$(A - z)w^{(n)} + Bw^{(n-1)} = 0, \qquad A \neq 0,$$

(v) $$w^{(n)} + Aw^{(n-k)} = 0, \qquad A \neq 0.$$

Since equations (iv) and (v) are independent of a, $\hat{a}$ they have TTRF's relative to *all* points. Aside from these trivial exceptions, then, an M-equation with a TTRF relative to two distinct finite points is a transform of Gauss' equation (19): If in (i), (ii), (iii) we transform[14] $\tilde{w} = w^{(n-2)}$, the new equations are transforms of (19) by Theorem 2.

By reasoning similar to that which led to Corollary 5 we arrive at

COROLLARY 6. *Necessary and sufficient conditions that an M-equation have a TTRF relative to three distinct finite points, but not all points, are that one of the points be the bisector of the line segment joining the other two and that the equation (except for a constant factor not zero) be the case of type* (i) *where $A = \hat{A}$ and the three points are a, $\hat{a}$, $(a + \hat{a})/2$.*

From the configuration thus necessary for any three of the points one can argue the impossibility of an M-equation with a TTRF relative to four distinct finite points, but not all points and hence state

COROLLARY 7. *If an M-equation has a TTRF relative to four distinct finite points, it has a TTRF relative to all points.*

6. **The second order case.** For any second order M-equation Theorem 2 gives a transformation to a GHE with $q = 1$, $p = 0$, 1, or 2. In the case $p = 2$, the transform is Gauss' equation. In the remaining cases $p = 0$ or 1, it is possible to transform to another well-known equation, namely *Whittaker's W_{km} equation*[15]

$$x^2\frac{d^2W}{dx^2} + \left(\tfrac{1}{4} - m^2 + kx - \frac{x^2}{4}\right)W = 0. \tag{26}$$

By Corollary 2 this equation has a TTRF relative to some point if and only if $k = 0$. If $k = 0$, Theorem 2 gives a transformation to a GHE with $q = 1$, $p = 0$. This transformation may be inverted, and any GHE with $q = 1$, $p = 0$, may be transformed to the W_{0m} equation. The constants of the transformation are given in Corollary 8.

The transformation

$$u = e^{-x/2}W$$

takes (26) into

$$x^2\frac{d^2u}{dx^2} + x^2\frac{du}{dx} + (\tfrac{1}{4} - m^2 + kx)u = 0. \tag{27}$$

By Corollary 2, (27) has a TTRF relative to $x = 0$, and Theorem 2 gives a transformation to a GHE with $q = p = 1$. Again, we may invert the transformation and transform any GHE with $q = p = 1$ to (27) and hence to (26). The last

[14] If $n = 1$ we understand $B = 0$ and transform $\tilde{w}' = w$.

[15] Whittaker and Watson, *A Course in Modern Analysis*, Cambridge, 1927, ch. 16.

two transformations may be replaced by their product; the constants of the resulting transformation are listed in

COROLLARY 8. *Any second order M-equation may be transformed to Gauss' equation* (19) *or Whittaker's equation* (26). *If* $s = t = 2$, *the transform is* (19) *and the transformation is given in Theorem* 2. *In the remaining cases the transform is* (26) *and the transformation is given in the following chart*:

Case	$\beta, \kappa, \gamma, \alpha_1, \rho_1$ given by	Transformation	k	m
$s = 2,\ t = 1$	(22), (23)	$x = -\beta\xi^{\kappa},$	$\alpha_1 - \dfrac{\rho_1}{2}$	$\dfrac{\rho_1 - 1}{2}$
$s = 1,\ t = 2$	(24), (25)	$w = \xi^{\gamma - \kappa\rho_1/2} e^{\beta\xi^{\kappa}/2} W$		
$s = 2,\ t = 0$	(22), (23)	$x = 4\beta^{1/2}\xi^{\kappa/2},$	0	$\rho_1 - 1$
$s = 0,\ t = 2$	(24), (25)	$w = \xi^{\gamma + \kappa(1 - 2\rho_1)/4} W$		

Appendix. Suppose a DE with a TTRF is multiplied through by some function of z, say $1 + (z - a) + (z - a)^2$. This would not affect the solutions, but the DE would no longer have a TTRF! The apparent paradox lies in our definition of the recurrence formula so that it depends on the form of the DE rather than on the solutions. It would be found that while the new recurrence formula had more than two terms, it could be obtained by iteration of the TTRF.

The method of solving linear DE's by finding a recurrence formula for the coefficients in a power series solution is widely applied because of its generality and simplicity. Our definition of the recurrence formula is in accord with this procedure: the formula is obtained from the DE and not from the solutions, and the above difficulty is actually encountered.

The question now arises, given any DE

$$\sum_{j=0}^{n} P_j(z) w^{(j)} = 0. \tag{28}$$

does there exist a suitable function $f(z)$ which when multiplied into the DE yields a DE with a TTRF?—in other words, such that $P_j(z)f(z) \equiv p_j(z)$, where the $p_j(z)$ are of type (11)? While the answer in general is rather obvious, we state it explicitly for the following practically important case: Suppose equation (28) is "normalized" so that $P_n(z) \equiv 1$. By Corollary 2 it can then be converted into an M-equation through multiplication by some $f(z)$ if and only if the $P_j(z)$ are rational functions of the structure

$$P_j(z) = \frac{S_j - T_j(z - a)^{\lambda}}{[S_n - T_n(z - a)^{\lambda}](z - a)^{n-j}}, \tag{29}$$

for some number a, where $|S_n| + |T_n| \neq 0$, $S_i T_k \neq 0$ for some i, k. If (29) is satisfied, a suitable function $f(z)$ for the normalized form is then $S_n - T_n(z - a)^{\lambda}$.

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A NUMERICAL METHOD OF SOLVING INTEGRAL EQUATIONS IN TWO INDEPENDENT VARIABLES[1]

By Philip C. Magnusson

1. Introduction

The linear inhomogeneous integral equation of the second kind[2] is a convenient means of expressing numerous physical problems, and for that reason considerable attention has been given recently to methods of solving the equation. In practically all investigations to date, consideration has been limited to equations in one independent variable; although in many of the physical applications, notably those of interreflections of light among perfectly diffusing surfaces and skin effect in electrical conductors, the problem is inherently one of two independent variables. In this investigation two-variable problems are investigated.

The equation of interreflections in two variables may be written as follows:

$$L(x, y) = L_0(x, y) + \rho(x, y) \iint K(x, y; s, t)L(s, t)ds\, dt. \tag{1}$$

This equation expresses the luminosity, or quantity of light emitted per unit area, at point (x, y) on a "perfectly diffusing surface", when an initial luminosity $(L_0(x, y))$ is produced by external means. The diffuse reflection factor at point (x, y) is $\rho(x, y)$. The kernel is

$$K = \frac{\cos \theta_i \cos \theta_r}{\pi r^2} \tag{2}$$

where θ_i and θ_r are the angle of incidence and reflection respectively, and r is the distance between points (x, y) and (s, t). In the following discussion, the problem of interreflections between two parallel square surfaces, 2×2 with unit separation (Fig. 1) will be used as an illustration.

2. Method of Simultaneous Linear Algebraic Equations

Probably the best numerical method of solving integral equations in two independent variables is that of simultaneous linear algebraic equations. This method is similar to that developed by Crout[11] for homogeneous integral equations of the second kind with one independent variable, but the writer's plan was especially devised for two-variable problems.

The initial step in deriving the method is to substitute for the analytic integral of the integral equation an algebraic approximation consisting of a sum of values of the product function at discrete points, weighted according to Simpson's or Cotes' Rules. Thus,

$$L(x, y) = L_0(x, y) + \rho \frac{1}{\Sigma n_i} [n_1 K(x, y; 1)L_1 + n_2 K(x, y; 2)L_2 + \cdots]. \tag{3}$$

By replacing the x's and the y's in L_0, and in the kernels by the coordinates of L_1, an algebraic equation for L_1 in terms of itself and $(n-1)$ other discrete values of L will result:

$$L_1 = L_0 + \rho \frac{1}{\Sigma n_i} (n_1 K_{11} L_1 + n_2 K_{12} L_2 + n_3 K_{13} L_3 + \cdots). \tag{4}$$

If corresponding equations are written for the other $(n-1)$ L's, a simultaneous set of n equations is obtained which may then be solved by any convenient method. It is assumed that parabolic interpolation between the points for which the luminosity has been solved will yield the luminosity at any intermediate point. This interpolation may be effected with interpolation tables,

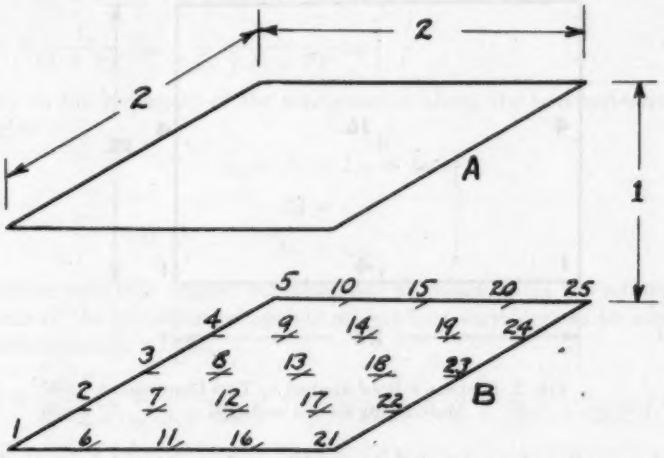

FIG. 1. Numbering Plan for Interreflection Problem
(Same numbering on upper as on lower plane)

such as those in Crout's paper, or the interpolating polynomial may be found. Lagrange's formula is convenient for this purpose:

$$\begin{aligned} F(x) = {} & \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} f_0 \\ & + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} f_1 + \cdots \\ & + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} f_n. \end{aligned} \tag{5}$$

Here f_0, f_1, etc. are the values of the function F at x_0, x_1, etc.

The problems of interreflections between two parallel, rectangular, plane surfaces with uniform reflection factors and uniform initial luminosities provides

a simple illustration of the method. Suppose discrete points are chosen as in Fig. 1. For the moment, only points, 1, 3, 5, 11, 13, 15, 21, 23, and 25 will be considered, and the lengths of the two sides will be taken as $2s$ and $2t$. Owing to symmetry, the luminosity function will be the same on both surfaces.

Simpson's Rule, which is commonly used to approximate a single-variable integral, may be extended to include a two-variable integral by forming the pattern shown in Fig. 2.[12] The kernel is

$$K = \frac{1}{\pi[1 + (x - \lambda)^2 + (y - \mu)^2]^2}. \tag{6}$$

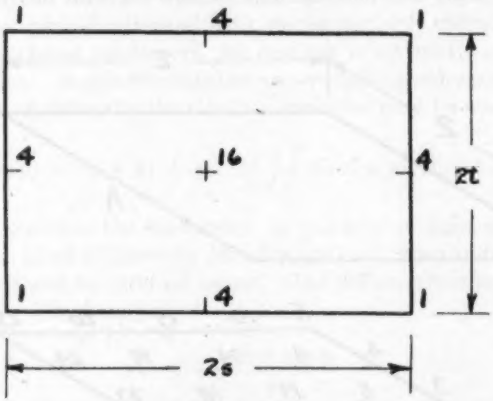

FIG. 2. Simpson's Rule Applied to Two Dimensions
Multiplying Factor = $4st/36$

Hence the equations for points 1, 3, 11 and 13 may be written down immediately:

$$L_1 = 1 + \frac{4st\rho}{36\pi}\left[\frac{1}{(1)^2}L_1 + 4\frac{1}{(1+s^2)^2}L_3 + \frac{1}{[1+(2s)^2]^2}L_5 + 4\frac{1}{(1+t^2)^2}L_{11}\right.$$
$$+ 16\frac{1}{(1+s^2+t^2)^2}L_{13} + 4\frac{1}{[1+(2s)^2+t^2]^2}L_{15} + \frac{1}{[1+(2t)^2]^2}L_{21}$$
$$\left. + 4\frac{1}{[1+s^2+(2t)^2]^2}L_{23} + \frac{1}{[1+(2s)^2+(2t)^2]^2}L_{25}\right],$$

$$L_3 = 1 + \frac{4st\rho}{36\pi}\left[\frac{1}{(1+s^2)^2}L_1 + 4\frac{1}{(1)^2}L_3 + \frac{1}{(1+s^2)^2}L_5 + 4\frac{1}{(1+s^2+t^2)^2}L_{11}\right.$$
$$+ 16\frac{1}{(1+t^2)^2}L_{13} + 4\frac{1}{(1+s^2+t^2)^2}L_{15} + \frac{1}{[1+s^2+(2t)^2]^2}L_{21}$$
$$\left. + 4\frac{1}{[1+(2t)^2]^2}L_{23} + \frac{1}{[1+s^2+(2t)^2]^2}L_{25}\right], \tag{7}$$

$$L_{11} = 1 + \frac{4st\rho}{36\pi}\left[\frac{1}{(1+t^2)^2}L_1 + 4\frac{1}{(1+s^2+t^2)^2}L_3 + \frac{1}{[1+(2s)^2+t^2)^2}L_5 + 4\frac{1}{(1)^2}L_{11} + 16\frac{1}{(1+s^2)^2}L_{13} + 4\frac{1}{[1+(2s)^2]^2}L_{15} + \frac{1}{(1+t^2)^2}L_{21} + 4\frac{1}{(1+s^2+t^2)^2}L_{23} + \frac{1}{[1+(2s)^2+t^2]^2}L_{25}\right],$$

$$L_{13} = 1 + \frac{4st\rho}{36\pi}\left[\frac{1}{(1+s^2+t^2)^2}L_1 + 4\frac{1}{(1+t^2)^2}L_3 + \frac{1}{(1+s^2+t^2)^2}L_5 + 4\frac{1}{(1+s^2)^2}L_{11} + 16\frac{1}{(1)^2}L_{13} + 4\frac{1}{(1+s^2)^2}L_{15} + \frac{1}{(1+s^2+t^2)^2}L_{21} + 4\frac{1}{(1+t^2)^2}L_{23} + \frac{1}{(1+s^2+t^2)^2}L_{25}\right].$$

Owing to the symmetry of the configuration about the two mid-axes of the rectangles:

$$\left.\begin{aligned} L_1 &= L_5 = L_{21} = L_{25}, \\ L_3 &= L_{23}, \\ L_{11} &= L_{15}. \end{aligned}\right\} \tag{8}$$

Therefore only four regular equations are necessary to fix the relationships. By means of the equalities just given, all but four variables can be eliminated from each equation. Thus,

$$\left.\begin{aligned} L_1 &= 1 + \frac{4st\rho}{36\pi}\left[\left(1 + \frac{1}{[1+(2s)^2]^2} + \frac{1}{[1+(2t)^2]^2} + \frac{1}{[1+(2s)^2+(2t)^2]^2}\right)L_1 + 4\left(\frac{1}{(1+s^2)^2} + \frac{1}{[1+s^2+(2t)^2]^2}\right)L_3 + 4\left(\frac{1}{(1+t^2)^2} + \frac{1}{[1+(2s)^2+t^2]^2}\right)L_{11} + 16\frac{1}{(1+s^2+t^2)^2}L_{13}\right], \\ L_3 &= 1 + \frac{4st\rho}{36\pi}\left[2\left(\frac{1}{(1+s^2)^2} + \frac{1}{[1+s^2+(2t)^2]^2}\right)L_1 + 4\left(1 + \frac{1}{[1+(2t)^2]^2}\right)L_3 + 8\frac{1}{(1+s^2+t^2)^2}L_{11} + 16\frac{1}{(1+t^2)^2}L_{13}\right], \\ L_{11} &= 1 + \frac{4st\rho}{36\pi}\left[2\left(\frac{1}{(1+t^2)^2} + \frac{1}{[1+(2s)^2+t^2]^2}\right)L_1 + 8\frac{1}{(1+s^2+t^2)^2}L_3 + 4\left(1 + \frac{1}{[1+(2s)^2]^2}\right)L_{11} + 16\frac{1}{(1+s^2)^2}L_{13}\right], \\ L_{13} &= 1 + \frac{4st\rho}{36\pi}\left[4\frac{1}{(1+s^2+t^2)^2}L_1 + 8\frac{1}{(1+t^2)^2}L_3 + 8\frac{1}{(1+s^2)^2}L_{11} + 16L_{13}\right]. \end{aligned}\right\} \tag{9}$$

3. Singularities and Near-Singularities

Instances are numerous in which a function is infinite at some point, yet has a finite integral over any area including that point. One such example is the kernel at the intersection of interreflecting surfaces. In other instances a function may rise to a high, yet finite "hump," which will be termed a near-singularity. The problem of interreflections between parallel planes, when worked with a few points, involves an illustration of the latter. The numerical integration methods assume that the value assigned to a function at one of the discrete points is representative of the function over a finite area surrounding it; hence the values to be used at singularities and near-singularities should be chosen so as to be "correct" when used in some other integral that is similar to the integral in the integral equation but which is subject to analytic check. Usually this integral is that of the kernel alone, taken over the entire physical surface. The example of interreflections between parallel plane surfaces (see Appendix), worked first on the basis of nine points with exact kernel values, then with the kernel values modified as just mentioned, and then on the basis of twenty-five points, shows the important gain in accuracy which is made by thus modifying the kernel.

4. Simplification by Adding Symmetry to Initial Luminosity

The resultant luminosity is a linear function of the initial luminosity in any interreflection problem. That is to say, the resultant luminosity produced by two sources (initial luminosity functions) is equal to the sum of the resultant luminosities produced by those sources separately. No such relation applies, however, for the addition of extra surfaces or for changes in the reflection factors. Thus the geometry of the surfaces, together with the reflection factors, determine the maximum degree of symmetry obtainable in a given problem; but if the initial luminosity exhibits less symmetry, additional initial luminosity may be introduced in a mathematical sense in such a way as to take full advantage of the symmetry of the configuration and thereby simplify the solution.

An example is that of two parallel, rectangular plane surfaces, with uniform initial luminosity on only one, but with uniform and equal reflection factors on both. If n independent points on each plane are to be solved for, solution of a set of $2n$ equations is apparently required. If however, an equal initial luminosity were assumed on the other plane, thereby making the initial luminosity as well as the fundamental geometry symmetrical about a hypothetical plane midway between the two reflecting planes, the solution then obtained for any point would be equal to the sum of the solutions for the corresponding points on the two planes in the original problem. This numerical relation between the two points can then be used to eliminate half the variables from the first n equations.

Let the subscript A indicate the plane with initial luminosity, and B indicate the plane without initial luminosity. Then

$$
\begin{aligned}
L_{1A} &= 1 + \rho(n_1K_{11}L_{1B} + n_2K_{12}L_{2B} + n_3K_{13}L_{3B} \cdots) \\
L_{2A} &= 1 + \rho(n_1K_{21}L_{1B} + n_2K_{22}L_{2B} + n_3K_{23}L_{3B} \cdots) \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
L_{1B} &= \rho(n_1K_{11}L_{1A} + n_2K_{12}L_{2A} + n_3K_{13}L_{3A} \cdots) \\
L_{2B} &= \rho(n_1K_{21}L_{1A} + n_2K_{22}L_{2A} + n_3K_{23}L_{3A} \cdots) \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
\tag{10}
$$

Let $L_{1C} = L_{1A} + L_{1B}$; $L_{2C} = L_{2A} + L_{2B}$; etc.

$$
\begin{aligned}
L_{1C} &= 1 + \rho(n_1K_{11}L_{1C} + n_2K_{12}L_{2C} + n_3K_{13}L_{3C} \cdots) \\
L_{2C} &= 1 + \rho(n_1K_{21}L_{1C} + n_2K_{22}L_{2C} + n_3K_{23}L_{3C} \cdots) \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
$$

After numerical values have been found for L_{1C} , L_{2C} , etc., these are substituted in the first n equations.

$$
\begin{aligned}
L_{1C} - L_{1B} &= 1 + \rho(n_1K_{11}L_{1B} + n_2K_{12}L_{2B} + n_3K_{13}L_{3B} \cdots) \\
L_{2C} - L_{2B} &= 1 + \rho(n_1K_{21}L_{1B} + n_2K_{22}L_{2B} + n_3K_{23}L_{3B} \cdots) \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
\tag{11}
$$

Thus the solution of a set of $2n$ equations is reduced to the solution of two sets of n equations each—a very marked saving in numerical work.

A generalization with respect to reflection factors is possible if the reflections take place entirely between two plane surfaces which have uniform (but not necessarily the same) reflection factors. Then the resultant luminosities are primarily functions of the geometric mean reflection factor. Assuming initial luminosity on one surface only, the resultant luminosity for that surface is the same for all combinations of reflection factors which have the same geometric mean. For the other surface, the result obtained with the geometric mean factor must be divided by this factor and multiplied by the true reflection factor of that surface. An example is given in the Appendix.

5. Recommended Procedure

The scope of problems that can be solved by this method is limited principally by the number of independent values which must be used to attain the required accuracy, and that number in turn is determined by the complexity of the unknown function. The exact nature of the unknown function, of course, cannot be evaluated until the problem has been worked, but inspection of the problem from a physical viewpoint will indicate whether discontinuities, abrupt changes, or undulations are to be expected. Presence of these phenomena tends to increase the necessary number of discrete points and equations, as does the requirement of a high degree of accuracy; whereas symmetry in the geometry has the opposite effect.

A problem in which a discontinuity occurs in the unknown function is that of interreflections between the floor and ceiling of a room in which part of the floor is covered with a dark rug but the remainder is highly reflective. The floor

luminosity would then be treated as two distinct functions, one continuous function to determine the luminosity on the rug, and a second continuous function to determine the luminosity on the bare section. To solve for the two functions, especially in view of the fact that one of the areas would be rather irregular, would obviously require a sharp increase in the number of discrete points over that required for a floor with uniform reflectivity. Symmetry in the geometry is a most powerful aid to simplification and is, fortunately, a property possessed to some degree by most common configurations. The need for checking the accuracy of the result is obvious. The most direct plan is to work the problem twice, using considerably more points in one solution than in the other, and then compare results.

Use of parabolic interpolation in both directions on a rectangular surface requires that values of the function be known at a minimum of nine discrete points, i.e., the four corners, the middle of each side, and the center of the rectangle. In an interreflection problem involving only rectangular surfaces, nine points on each surface—distributed as just stated—would provide a logical basis for a trial solution. If the problem were then reworked on the basis of twenty-five points on each surface, and if the results of the two solutions showed close agreement (values in the nine-point solution for the additional sixteen points would be found by interpolation), one could reasonably conclude that the second solution was accurate well within the discrepancy between the two solutions.

A modification of this procedure which saves some labor is to write the twenty-five-point equations only for the points used in the nine-point solution, and then express the sixteen added points in terms of the other nine by interpolation formulas. This process is illustrated in the interreflection problem in the Appendix.

6. Conclusions

Numerous physical problems are readily expressed by means of integral equations, and many of these—such as interreflections and skin effect—are normally two-variable problems. Analytic attacks on two-variable integral equations become hopelessly involved in all but the simplest cases, and a solution by means of the Cinema Integraph is extremely lengthy. Of the possible numerical plans, an algebraic solution in which the product integrals are approximated by weighted averages of the function values at discrete points appears to be the simplest and most direct method.

The outstanding advantage of the algebraic method is that it requires no apparatus. The disadvantage of solving for the unknown function at only a finite number of values of the variables is not nearly as serious as it might appear at first glance, since parabolic interpolation gives a very good approximation to any smooth function. The arithmetical labor of solving most practical problems without any simplifying assumptions would be tremendous, but if judicious use is made of such assumptions the problem can be made relatively easy. As shown by the examples in the Appendix, the solution of a two-variable integral

equation is not inherently an endless task, but on the contrary, a great many such problems of moderate complexity can be attacked systematically and solved with reasonable labor by the method of simultaneous algebraic equations.

APPENDIX

a. Uniform Initial Luminosity on Both Surfaces; Uniform and Equal Reflection Factors

Nine discrete points per surface (eighteen altogether) will serve as a good basis for a trial solution in this problem. Owing to the high degree of symmetry, only three of those points will be independent. (See Fig. 1)

$$L_{1A} = L_{1B} = L_{5A} = L_{5B} = L_{21A} = L_{21B} = L_{25A} = L_{25B},$$

$$L_{3A} = L_{3B} = L_{11A} = L_{11B} = L_{15A} = L_{15B} = L_{23A} = L_{23B},$$

$$L_{13A} = L_{13B}.$$

As noted in the main discussion, the kernel for this general problem is

$$K = \frac{1}{\pi[1 + (x - \lambda)^2 + (y - \mu)^2]^2}. \tag{6}$$

The solution will first be carried through without modifying the numerical kernel, in order that the effect of modifying it to better represent the kernel function may be shown later. The equations may be written down directly, using the numerical values of the kernel and the weighting plan for nine points on a rectangle. (See Fig. 2):

$$L_1 = 1 + \frac{\rho}{9\pi}[(1.000 + 0.040 + 0.040 + 0.012)L_1 + 4(0.250 + 0.250 + 0.028 + 0.028)L_3 + 16(0.111)L_{13}]$$

$$L_3 = 1 + \frac{\rho}{9\pi}[2(0.250 + 0.028)L_1 + 4(1.000 + 0.111 + 0.111 + 0.040)L_3 + 16(0.250)L_{13}]$$

$$L_{13} = 1 + \frac{\rho}{9\pi}[4(0.111)L_1 + 4 \times 4(0.250)L_3 + 16(1.000)L_{13}]$$

$$L_1 = 1 + \rho(0.0386L_1 + 0.0786L_3 + 0.0629L_{13})$$

$$L_3 = 1 + \rho(0.0197L_1 + 0.1788L_3 + 0.1413L_{13})$$

$$L_{13} = 1 + \rho(0.0157L_1 + 0.1413L_3 + 0.5652L_{13}).$$

The higher the reflection factor, the more pronounced is the interreflection effect. The reflection factor will be assumed to be unity in this example, although that condition can be only approached in practice. The last set of equations then yields the following results:

$$L_1 = 1.378, \quad L_3 = 1.763, \quad L_{13} = 2.930.$$

In this numerical method, the approximation may be poor if the kernel contains a near-singularity, and examination of this kernel shows that the latter condition exists if the approximation is based on so few points. The analytic values of the integral of the kernel alone for points 1, 3, and 13 are as follows:

$$1, \int K d\sigma = \frac{2 \cdot 2}{2\pi\sqrt{1+2^2}} \tan^{-1} \frac{2}{\sqrt{1+2^2}} = \frac{2}{\pi\sqrt{5}} \tan^{-1} \frac{2}{\sqrt{5}} = 0.207$$

$$3, \int K d\sigma = \frac{2 \cdot 2}{2\pi\sqrt{1+2^2}} \tan^{-1} \frac{1}{\sqrt{1+2^2}} + \frac{2 \cdot 1}{2\pi\sqrt{1+1}} \tan^{-1} \frac{2}{\sqrt{1+1}} = 0.334$$

$$13, \int K d\sigma = \frac{8 \cdot 1}{2\pi\sqrt{1+1}} \tan^{-1} \frac{1}{\sqrt{1+1}} = 0.554.$$

The approximated values are

$$1, \int K d\sigma = 0.0386 + 0.0786 + 0.0629 = 0.1801$$

$$3, \int K d\sigma = 0.0197 + 0.1788 + 0.1413 = 0.3398$$

$$13, \int K d\sigma = 0.0157 + 0.1413 + 0.5652 = 0.7222.$$

Since the near-singularity is at $x = \lambda$, $y = \mu$, it appears from the foregoing results that the equation for L_{13} weights the L_{13} term much too heavily, and the equation for L_1 weights the L_1 term somewhat too lightly. This may be corrected by changing those coefficients so as to make the approximated kernel-integral equal to the analytic value, or

$$L_1 = 1 + \rho(0.0656L_1 + 0.0786L_3 + 0.0629L_{13})$$

$$L_3 = 1 + \rho(0.0197L_1 + 0.1788L_3 + 0.1413L_{13})$$

$$L_{13} = 1 + \rho(0.0157L_1 + 0.1413L_3 + 0.3972L_{13}).$$

These equations yield the following values when $\rho = 1$:

$$L_1 = 1.344, \qquad L_3 = 1.603, \qquad L_{13} = 2.065.$$

The differences between this solution and that obtained with the unmodified kernel are marked indeed.

Inasmuch as the variation of the resultant luminosity over the surface is not extreme, and the modified approximation of the kernel is exact when integrated with a constant, the approximation of the product integral in the foregoing solution is probably good. A check may be made by setting up the equations in terms of twenty-five points (six of them independent), and reducing them to three independent points by interpolation formulae.

Cotes' Rule for single-variable approximate integration by the use of five

discrete points is expanded into a two-dimensional pattern similar to Fig. 2 for use here. The single-variable form is:

$$\int_{x_1}^{x_5} f(x)\,dx = (1/22.5)h(7f_1 + 32f_2 + 12f_3 + 32f_4 + 7f_5)$$

Here f_1, f_2, etc. are the values of f at equally spaced points, h distance apart. The resulting mesh is shown in Fig. 3.

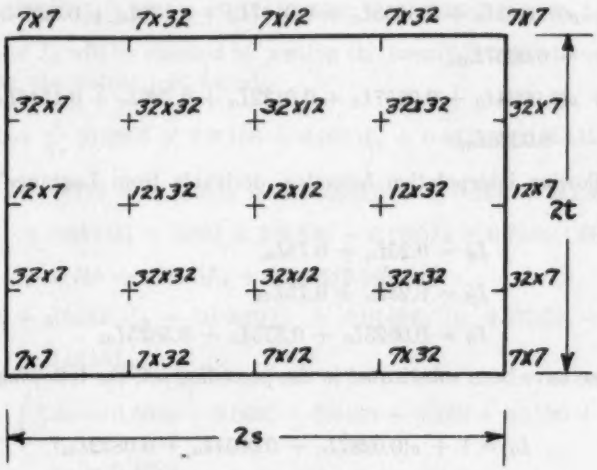

FIG. 3. Two Dimensional Mesh for Cotes' Rule
Multiplying Factor = $4st/(90)^2$

$$L_1 = 1 + \frac{\rho}{4\pi}[0.0968(1.000 + 0.040 + 0.040 + 0.012)L_1 + 0.0425 \times 2(0.640 + 0.095 + 0.036 + 0.019)L_2 + 0.1659(0.250 + 0.250 + 0.028 + 0.028)L_3 + 2.023(0.445 + 0.082 + 0.082 + 0.033)L_7 + 0.7586 \times 2(0.198 + 0.055)L_8 + (0.2844(0.111)L_{13}]$$

$$L_3 = 1 + \frac{\rho}{4\pi}[0.0968 \times 2(0.250 + 0.028)L_1 + 0.4425 \times 2(0.640 + 0.198 + 0.055 + 0.036)L_2 + 0.1659(1.000 + 0.111 + 0.111 + 0.040)L_3 + 2.023 \times 2(0.445 + 0.082)L_7 + 0.7586(0.640 + 0.198 + 0.198 + 0.095)L_8 + 0.2844(0.250)L_{13}]$$

$$L_{13} = 1 + \frac{\rho}{4\pi}[0.0068 \times 4(0.111)L_1 + 0.4425 \times 8(0.198)L_2 + 0.1659 \times 4(0.250)L_3 + 2.023 \times 4(0.445)L_7 + 0.7586 \times 4(0.640)L_8 + 0.2844(1.000)L_{13}]$$

$$L_1 = 1 + \rho(0.0084L_1 + 0.0555L_2 + 0.0073L_3 + 0.103L_7 + 0.0305L_8 + 0.0023L_{13})$$

$$L_3 = 1 + \rho(0.0043L_1 + 0.0655L_2 + 0.0167L_3 + 0.170L_7 + 0.0687L_8 + 0.0057L_{13})$$

$$L_{13} = 1 + \rho(0.0034L_1 + 0.0557L_2 + 0.0132L_3 + 0.286L_7 + 0.1545L_8 + 0.0226L_{13})$$

The following interpolation formulae, derivable from Lagrange's formula, are used:

$$L_2 = 0.25L_1 + 0.75L_3$$
$$L_8 = 0.25L_3 + 0.75L_{13}$$
$$L_7 = 0.0625L_1 + 0.375L_3 + 0.5625L_{13}$$

When these have been substituted in the preceding set, the following equations result:

$$L_1 = 1 + \rho(0.0287L_1 + 0.0951L_3 + 0.0832L_{13})$$
$$L_3 = 1 + \rho(0.0313L_1 + 0.1468L_3 + 0.1528L_{13})$$
$$L_{13} = 1 + \rho(0.0352L_1 + 0.2006L_3 + 0.2996L_{13}).$$

The approximated integrals of the kernel alone are

$$1, \int K d\sigma = 0.0287 + 0.0951 + 0.0832 = 0.2070$$

$$3, \int K d\sigma = 0.0313 + 0.1468 + 0.1528 = 0.3309$$

$$13, \int K d\sigma = 0.0352 + 0.2006 + 0.2996 = 0.5354$$

These are much closer to the correct values than are the nine-point approximation values. The equations are then adjusted to make the integral of the kernel alone exact:

$$L_1 = 1 + \rho(0.0287L_1 + 0.0951L_3 + 0.0832L_{13})$$
$$L_3 = 1 + \rho(0.0313L_1 + 0.1498L_3 + 0.1528L_{13})$$
$$L_{13} = 1 + \rho(0.0352L_1 + 0.2006L_3 + 0.3182L_{13}).$$

when $\rho = 1$, the following results are obtained:

$$L_1 = 1.360, \qquad L_3 = 1.585, \qquad L_{13} = 2.000.$$

The values for the intermediate points may then be found by the Lagrange interpolation formulae:

$$L_2 = (0.25)(1.360) + (0.75)(1.585) = 1.530$$

$$L_8 = (0.25)(1.585) + (0.75)(2.000) = 1.896$$

$$L_7 = (0.0625)(1.360) + (0.375)(1.585) + (0.5625)(2.000) = 1.805.$$

The value of L_8 will be checked by writing the twenty-five point equation and substituting the values just found.

$$L_8 = 1 + \frac{\rho}{4\pi}\,[0.0968 \times 2(0.198 + 0.055)L_1 + 0.4425 \times 2(0.445 + 0.250 + 0.111 + 0.055)L_2 + 0.1659(0.640 + 0.198 + 0.198 + 0.095)L_3 + 2.023 \times 2(0.640 + 0.198)L_7 + 0.7586(1.000 + 0.445 + 0.445 + 0.250)L_8 + 0.2844(0.640)L_{13}]$$

$$L_8 = 1 + \rho(0.0039L_1 + 0.0607L_2 + 0.0149L_3 + 0.270L_7 + 0.1290L_8 + 0.0145L_{13})$$

$$\text{Approx.} \int K d\sigma = 0.0039 + 0.0607 + 0.0149 + 0.270 + 0.1290 + 0.0145 = 0.4930$$

$$\text{Analytic} \int K d\sigma = 0.4982$$

Raise the coefficient of L_8 in the equation by 0.005. After substituting values from (47) and (48) for the other L's,

$$L_8 = 1.890.$$

The close agreement between the two values of L_8 indicates that nine points are adequate to represent the resultant luminosity.

b. Uniform Initial Luminosity on "A" Surface Only; Uniform and Equal Reflection Factors

In this situation the procedure of adding symmetry to the initial luminosity may be used to advantage. Then

$$L_{1A} = 1 + \rho(0.0287L_{1B} + 0.0951L_{3B} + 0.0832L_{13B})$$

$$L_{3A} = 1 + \rho(0.0313L_{1B} + 0.1498L_{3B} + 0.1528L_{13B})$$

$$L_{13A} = 1 + \rho(0.0352L_{1B} + 0.2006L_{3B} + 0.3182L_{13B})$$

$$L_{1B} = \rho(0.0287L_{1A} + 0.0951L_{3A} + 0.0832L_{13A})$$

$$L_{3B} = \rho(0.0313L_{1A} + 0.1498L_{3A} + 0.1528L_{13A})$$

$$L_{13B} = \rho(0.0352L_{1A} + 0.2006L_{3A} + 0.3182L_{13A})$$

Following the plan outlined in the main discussion, let

$$L_{nC} = L_{nA} + L_{nB}.$$

Then the L_C's (for $\rho = 0.9$) are

$$L_{1C} = 1 + 0.0258L_{1C} + 0.0856L_{3C} + 0.0749L_{13C}$$

$$L_{3C} = 1 + 0.0282L_{1C} + 0.1350L_{3C} + 0.1375L_{13C}$$

$$L_{13C} = 1 + 0.0317L_{1C} + 0.1805L_{3C} + 0.2864L_{13C}$$

The following values are obtained:

$$L_{1C} = 1.300, \qquad L_{3C} = 1.490, \qquad L_{13C} = 1.835.$$

Then

$$1.300 - L_{1B} = 1 + 0.0258L_{1B} + 0.0856L_{3B} + 0.0749L_{13B}$$

$$1.490 - L_{3B} = 1 + 0.0282L_{1B} + 0.1350L_{3B} + 0.1375L_{13B}$$

$$1.835 - L_{13B} = 1 + 0.0317L_{1B} + 0.1805L_{3B} + 0.2864L_{13B}$$

$$L_{1B} = 0.220, \qquad L_{3B} = 0.353, \qquad L_{13B} = 0.595,$$

$$L_{1A} = 1.300 - 0.220 = 1.080.$$

$$L_{3A} = 1.490 - 0.353 = 1.137$$

$$L_{13A} = 1.835 - 0.595 = 1.240$$

c. Uniform Initial Luminosity on "A" Surface Only; Uniform but Unequal Reflection Factors

Let $\rho_A = 1.0, \qquad \rho_B = 0.81.$

Then $\rho_M = \sqrt{(1.0)(0.81)} = 0.9.$

Since the geometric mean reflection factor is the same as that used in *b*, the results of that solution may be used here, subject to modification as described in Section *c*.

$$L_{1A} = 1.080, \qquad L_{3A} = 0.137, \qquad L_{13A} = 1.240.$$

$$L_{1B} = 0.220(0.81/0.9) = 0.198$$

$$L_{3B} = 0.353(0.81/0.9) = 0.318$$

$$L_{13B} = 0.595(0.81/0.9) = 0.536$$

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ON THE FUNCTION $H(m, a, x) = \exp(-ix)F(m + 1 - ia, 2m + 2; 2ix)$

By Arnold N. Lowan and William Horenstein[1]

Confluent hypergeometric functions occur in numerous physical problems. They appear in the solutions of Laplace's equation and the equations of wave motion and of diffusion when the physical problem involves knife edges, parabolic cylinders, paraboloids of revolution, general ellipsoids, etc. Of particular importance to the quantum physicist is their occurrence as the solutions of the radial part of Schrödinger's equation for a Coulomb field and continuous values of the energy. Thus they are needed for the calculation of the scattering of charged particles from atomic nuclei, or of their binding by nuclei. In a more general form they also occur in the problem of dissociation of molecules into atomic ions (and, of course, in the converse case of the formation of molecules).

The radial component of the wave function for two particles with the charges Z_1e and $-Z_2e$ respectively, is:

$$R = e^{-i\lambda r}r^m F(m + 1 - ia, 2m + 2; 2i\lambda r)$$

with

$$\lambda = \frac{(2\mu E)^{\frac{1}{2}}}{h} \quad \text{and} \quad a = \frac{Z_1 Z_2 e^2(\mu/2E)^{\frac{1}{2}}}{h}$$

We tabulate here the functions

$$H(m, a, x) = e^{-ix}F(m + 1 - ia, 2m + 2; 2ix) \text{ and } \frac{d}{dx}H(m, a, x)$$

i.e. essentially $r^{-m}R$

Fundamental Properties

Consider the Hypergeometric series

$$(1) \qquad F(\alpha, \beta, \gamma; x) = 1 + \frac{\alpha\beta}{1\gamma}x + \frac{\alpha(\alpha+1)}{1.2}\frac{\beta(\beta+1)}{\gamma(\gamma+1)}x^2 \cdots$$

which is one of the solutions of Gauss' Hypergeometric Differential Equation

$$(2) \qquad x(1-x)\frac{d^2y}{dx^2} + \{\gamma - (\alpha + \beta + 1)x\}\frac{dy}{dx} - \alpha\beta y = 0$$

If in (1) and (2) we make the substitution $x = \frac{u}{\beta}$ and allow β to approach ∞, we obtain

$$(3) \quad F(\alpha, \gamma; u) = 1 + \frac{\alpha}{\gamma}\frac{u}{1!} + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\frac{u^2}{2!} \cdots + \frac{\alpha(\alpha+1)\cdots(\alpha+n)}{\gamma(\gamma+1)\cdots(\gamma+n)}\frac{u^{n+1}}{(n+1)!} \cdots$$

[1] Members of the Mathematical Tables Project, operated by the Work Projects Administration, New York City, under the sponsorship of Dr. Lyman J. Briggs, Director, National Bureau of Standards.

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and

$$u\frac{d^2y}{du^2} + (\gamma - u)\frac{dy}{du} - \alpha y = 0 \tag{4}$$

Let $u = 2ix$; $\alpha = m + 1 - ia$; $\gamma = 2m + 2$, then (4) becomes

$$x\frac{d^2y}{dx^2} + (2m + 2 - 2ix)\frac{dy}{dx} - 2i(m + 1 - ia)y = 0 \tag{5}$$

If we assume a solution of (5) of the form $y = e^{ix}H(m, a, x)$, it is readily seen the function

$$H(m, a, x) = e^{-ix}F(m + 1 - ia, 2m + 2; 2ix) \tag{6}$$

satisfies the differential equation

$$x\frac{d^2H}{dx^2} + (2m + 2)\frac{dH}{dx} + (x - 2a)H = 0 \tag{7}$$

For the sake of subsequent developments it will be convenient to derive an integral representation for the function $H(m, a, x)$. Referring to (3), let

$$U_n = \frac{\alpha(\alpha + 1)\cdots(\alpha + n)}{\gamma(\gamma + 1)\cdots(\gamma + n)} = \frac{\Gamma(\alpha + n + 1)\Gamma(\gamma)}{\Gamma(\gamma + n + 1)\Gamma(\alpha)}$$

In the well known relation

$$\frac{\Gamma(\theta)\Gamma(\phi)}{\Gamma(\theta + \phi)} = \beta(\theta, \phi) = \int_0^1 x^{\theta-1}(1 - x)^{\phi-1}\,dx,$$

let $\theta = \alpha + n + 1$, $\theta + \varphi = \gamma + n + 1$, so that $\varphi = \gamma - \alpha$ then

$$\frac{\Gamma(\alpha + n + 1)}{\Gamma(\gamma + n + 1)} = \frac{1}{\Gamma(\gamma - \alpha)}\int_0^1 x^{n+\alpha}(1-x)^{\gamma-\alpha-1}\,dx,$$

and

$$U_n = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma - \alpha)}\int_0^1 x^{n+1}x^{\alpha-1}(1-x)^{\gamma-\alpha-1}\,dx$$

It follows that

$$F(\alpha, \gamma; u) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma - \alpha)}\int_0^1 \left(1 + \frac{xu}{1!} + \frac{x^2u^2}{2!} + \cdots\right)x^{\alpha-1}(1 - x)^{\gamma-\alpha-1}\,dx$$

hence

$$F(\alpha, \gamma; u) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma - \alpha)}\int_0^1 e^{xu}x^{\alpha-1}(1 - x)^{\gamma-\alpha-1}\,dx \tag{8}$$

In view of the last result the function $H(m, a, x)$ defined by (6) becomes

$$H(m, a, x) = \frac{e^{-ix}\Gamma(2m + 2)}{\Gamma(m + 1 - ia)\Gamma(m + 1 + ia)}\int_0^1 \xi^{m-ia}(1 - \xi)^{m+ia}e^{2ix\xi}\,d\xi$$

With the aid of well known properties of the Gamma Function, it may be shown that

$$\Gamma(m+1-ia)\Gamma(m+1+ia) = \prod_{k=0}^{m} \{(m-k)^2 + a^2\} \frac{\pi}{a \sinh \pi a}$$

hence

$$H(m, a, x) = \frac{e^{-ix}\Gamma(2m+2)a \sinh \pi a}{\prod_{k=0}^{m} \{(m-k)^2 + a^2\}\pi} \int_0^1 \xi^{m-ia}(1-\xi)^{m+ia} e^{2ix\xi}\, d\xi$$

If in the last expression we make the substitution $\xi = \frac{\phi+1}{2}$ the second member becomes after some obvious transformations

$$(9) \quad H(m, a, x) = \frac{2^{-2m}\Gamma(2m+2)a \sinh \pi a}{\prod_{k=0}^{m} \{(m-k)^2 + a^2\}\pi} \int_0^1 (1-\phi^2)^m \cos\left\{x\phi + a \log \frac{1-\phi}{1+\phi}\right\} d\phi$$

For simplicity let us write the last expression as

$$(10) \qquad H(m, a, x) = T(m, a) \int_0^1 (1-\phi^2)^m \cos \omega\, d\phi$$

where

$$(11) \quad T(m, a) \equiv \frac{2^{-2m}\Gamma(2m+2)a \sinh \pi a}{\prod_{k=0}^{m} \{(m-k)^2 + a^2\}\pi}, \qquad \omega \equiv x\phi + a \log \frac{1-\phi}{1+\phi}$$

then

$$(12) \quad \begin{aligned} H(m+1, a, x) &= T(m+1, a) \int_0^1 (1-\phi^2)^{m+1} \cos \omega\, d\phi \\ &= T(m+1, a) \int_0^1 (1-\phi^2)^m \cos \omega\, d\phi - T(m+1, a) \int_0^1 (1-\phi^2)^m \phi^2 \cos \omega\, d\phi \end{aligned}$$

Integration by parts leads to

$$(13) \quad \begin{aligned} H(m+1, a, x) = T(m+1, a) &\int_0^1 (1-\phi^2)^m \cos \omega\, d\phi \\ &- \frac{2aT(m+1, a)}{2m+2} \int_0^1 (1-\phi^2)^m \phi \sin \omega\, d\phi \\ &+ \frac{xT(m+1, a)}{2m+2} \int_0^1 \phi(1-\phi^2)^{m+1} \sin \omega\, d\phi \\ &- \frac{T(m+1, a)}{2m+2} \int_0^1 (1-\phi^2)^{m+1} \cos \omega\, d\phi \end{aligned}$$

Let

$$I(m, a, x) = \int_0^1 (1 - \phi^2)^m \cos \omega \, d\phi \tag{14}$$

Whence

$$\frac{d}{dx} I(m, a, x) = - \int_0^1 \phi(1 - \phi^2)^m \sin \omega \, d\phi \tag{15}$$

Rewrite (13) with the aid of (11) in the form

$$\begin{aligned} H(m + 1, a, x) = {} & \frac{(2m + 2)(2m + 3)}{4\{(m + 1)^2 + a^2\}} T(m, a) \int_0^1 (1 - \phi^2)^m \cos \omega \, d\phi \\ & - \frac{2a(2m + 3)}{4\{(m + 1)^2 + a^2\}} T(m, a) \int_0^1 (1 - \phi^2)^m \phi \sin \omega \, d\phi \\ & + \frac{xT(m + 1, a)}{2m + 2} \int_0^1 (1 - \phi^2)^{m+1} \phi \sin \omega \, d\phi \\ & - \frac{T(m + 1, a)}{2m + 2} \int_0^1 (1 - \phi^2)^{m+1} \cos \omega \, d\phi \end{aligned} \tag{16}$$

Making use of (10) and (15) we get

$$\begin{aligned} H(m + 1, a, x) = {} & \frac{(m + 1)^2}{(m + 1)^2 + a^2} H(m, a, x) + \frac{a(m + 1)}{(m + 1)^2 + a^2} \frac{d}{dx} H(m, a, x) \\ & - \frac{x}{2m + 3} \frac{d}{dx} H(m + 1, a, x) \end{aligned} \tag{17}$$

The differential equation (7) may be written as

$$x \frac{d^2}{dx^2} H(m, a, x) + (2m + 2) \frac{d}{dx} H(m, a, x) + (x - 2a) H(m, a, x) = 0 \tag{18}$$

Whence

$$\begin{aligned} x \frac{d^2}{dx^2} H(m + 1, a, x) + (2m + 4) \frac{d}{dx} H(m + 1, a, x) \\ + (x - 2a) H(m + 1, a, x) = 0 \end{aligned} \tag{19}$$

If we differentiate (17) with respect to x and eliminate $\frac{d^2}{dx^2} H(m, a, x)$ and $\frac{d^2}{dx^2} H(m + 1, a, x)$ with the aid of (18) and (19), then

$$\frac{d}{dx} H(m, a, x) = \frac{a}{m + 1} H(m, a, x) - \frac{x\{(m + 1)^2 + a^2\}}{(m + 1)^2 (2m + 3)} H(m + 1, a, x) \tag{20}$$

Additional Recursion Formulae will be obtained below.

If in (9) we set $a = 0$ and make use of the identity[2]

$$J_{m+\frac{1}{2}}(x) = \frac{1}{\sqrt{\pi}\,\Gamma(m+1)}\left(\frac{x}{2}\right)^{m+\frac{1}{2}}\int_{-1}^{+1}(1-\phi^2)^m\cos(x\phi)\,d\phi$$

we ultimately obtain

$$H(m, 0, x) = \frac{(2m+1)!}{m!(2x)^m}\sqrt{\frac{\pi}{2x}}\,J_{m+\frac{1}{2}}(x) \tag{21}$$

This relation will be used in subsequent developments.

Operational Derivation of Recurrence Formulae

The recurrence formulae satisfied by $H(m, a, x)$ may be conveniently derived from the "Laplace Transform" of the function $h(m, a; x) = x^{2m+1}H(m, a, x)$. Let

$$h^*(m, a; p) = \int_0^\infty e^{-px}h(m, a; x)\,dx = L\{h(m, a; x)\} \qquad \text{(say)} \tag{22}$$

where p is a complex variable whose real part is positive.

The derivation of the explicit expression for $h^*(m, a; p)$ is based on the following properties[3] of the Laplace Operator L defined above.

$$L\left\{\frac{d}{dx}h(m, a; x)\right\} = pL\left\{h(m, a; x)\right\} - h(m, a; 0) = ph^*(m, a; p) \tag{23}$$

$$\begin{aligned} L\left\{\frac{d^2h}{dx^2}\right\} &= p^2L\{h(m, a; x)\} - \left\{\frac{\partial}{\partial x}h(m, a; x)\right\}_{x=0} - ph(m, a; 0) \\ &= p^2h^*(m, a; p) \end{aligned} \tag{24}$$

$$L\{xh(m, a; x)\} = -\frac{d}{dp}h^*(m, a; p) \tag{25}$$

$$L\left\{x\frac{\partial^2}{\partial x^2}h(m, a; x)\right\} = -\frac{d}{dp}(p^2h^*) + h(m, a; 0) = -\frac{d}{dp}(p^2h^*) \tag{26}$$

It is readily verified that $h(m, a; x)$ satisfies the differential equation

$$xh'' - 2mh' + (x - 2a)h = 0 \tag{27}$$

If we multiply (27) by e^{-px}, integrate between 0 and ∞ and take into account relations (22–26), we ultimately get

$$\left\{\frac{d}{dp} + \frac{2p + 2mp + 2a}{p^2 + 1}\right\}h^*(m, a; p) = 0 \tag{28}$$

[2] Gray, Mathews and Macrobert, Bessel Functions, p. 46, eq. (6).

[3] Van der Pohl, Phil. Mag., 1929, p. 861, S.7.
Van der Pohl, Phil. Mag., 1929, p. 1153, S.7.

whence

$$(29) \qquad h^* = \int_0^\infty e^{-px} x^{2m+1} H(m, a, x)\, dx = \frac{C}{(p^2+1)^{m+1} e^{2a \arctan p}}$$

In order to determine the constant of integration C, set $a = 0$ and replace $H(m, 0, x)$ by its expression from (21)

This yields

$$(30) \qquad C = (1+p^2)^{m+1} \frac{(2m+1)!\sqrt{\pi}}{m!\, 2^{m+\frac{1}{2}}} \int_0^\infty e^{-px} x^{m+\frac{1}{2}} J_{m+\frac{1}{2}}(x)\, dx$$

With the aid of the identity[4]

$$(31) \qquad \int_0^\infty e^{-px} x^{m+\frac{1}{2}} J_{m+\frac{1}{2}}(x)\, dx = \frac{1}{\sqrt{\pi}} \Gamma(m+1) 2^{m+\frac{1}{2}} (p^2+1)^{-(m+1)}$$

equation (30) ultimately yields

$$C = (2m+1)!$$

so that

$$(32) \qquad h^*(m, a; p) = \frac{(2m+1)!}{(1+p^2)^{m+1} e^{2a \arctan p}} = L\{x^{(2m+1)} H(m, a, x)\}$$

In view of (23) and (25)

$$(33) \qquad ph^* = \frac{(2m+1)!\, p}{(1+p^2)^{m+1} e^{2a \arctan p}} = L\left\{\frac{d}{dx} x^{2m+1} H(m, a, x)\right\}$$

and $$\frac{d}{dp} h^*(m, a; p) = \frac{d}{dp}\left\{\frac{(2m+1)!}{(1+p^2)^{m+1} e^{2a \arctan p}}\right\} = -L\{x^{2m+2} H(m, a, x)\}$$

whence

$$(34) \qquad \frac{(2m+1)!\, 2a}{(1+p^2)^{m+2} e^{2a \arctan p}} + \frac{(2m+2)!\, p}{(1+p^2)^{m+2} e^{2a \arctan p}} = L\{x^{2m+2} H(m, a, x)\}$$

Substituting (32) and (33) in (34) yields the Recursion Formula

$$(35) \qquad \frac{d}{dx} H(m+1, a, x) = \frac{2m+3}{x} H(m, a, x) - \frac{(m+1)(2m+3) + ax}{(m+1)x} H(m+1, a, x)$$

In the differential equation {see (7)}

$$x \frac{d^2}{dx^2} H(m, a, x) + (2m+2) \frac{d}{dx} H(m, a, x) + (x - 2a) H(m, a, x) = 0$$

replace m by $m+1$, then

$$x \frac{d^2}{dx^2} H(m+1, a, x) + (2m+4) \frac{d}{dx} H(m+1, a, x) + (x - 2a) H(m+1, a, x) = 0$$

[4] Gray, Mathews and Macrobert, Bessel Functions, p. 76, ex. 10.

If we differentiate equation (35) with respect to x and eliminate

$$\frac{d^2}{dx^2}H(m+1,a,x)$$

with the aid of the last expression we obtain the Recursion Formula

$$(2m+3)(m+1)\frac{d}{dx}H(m,a,x) - \frac{(2m+3)(m+1)}{x}H(m,a,x)$$

$$+ (m+1-ax)\frac{d}{dx}H(m+1,a,x) \tag{36}$$

$$+ \frac{(2m+3+x^2-2ax)(m+1)}{x}H(m+1,a,x) = 0$$

Elimination of $\frac{d}{dx}H(m+1,a,x)$ between (35) and (36) results in the Recursion Formula

$$\frac{d}{dx}H(m,a,x) = \frac{a}{m+1}H(m,a,x) - \frac{x\{(m+1)^2+a^2\}}{(2m+3)(m+1)^2}H(m+1,a,x) \tag{37}$$

From (35) and (37) it is possible to derive the Recursion Formula

$$H(m+1,a,x) = \frac{(m+1)^2}{(m+1)^2+a^2}H(m,a,x)$$

$$+ \frac{a(m+1)}{(m+1)^2+a^2}\frac{d}{dx}H(m,a,x) - \frac{x}{2m+3}\frac{d}{dx}H(m+1,a,x) \tag{38}$$

Finally, replacing m by $m+1$ in (37) and eliminating $\frac{d}{dx}H(m+1,a,x)$ by means of (35), we obtain the Recursion Formula

$$x^2(m+1)\{(m+2)^2+a^2\}H(m+2,a,x)$$

$$-(m+2)(2m+3)(2m+5)\{(m+1)(m+2)+ax\}H(m+1,a,x) \tag{39}$$

$$+(m+1)(m+2)^2(2m+3)(2m+5)H(m,a,x) = 0$$

Method of Computation

Assuming a power series solution of the differential equation (7) of the form

$$H(m,a,x) = \sum_0^\infty c_k\frac{x^k}{k!} = \sum_0^\infty c_k U_k = \sum_0^\infty \xi_k \quad \text{(say)} \tag{40}$$

we obtain the recursion formula

$$(2m+2+k)c_{k+1} - 2ac_k + kc_{k-1} = 0 \tag{41}$$

From

$$H(m,a,x) = e^{-ix}F(m+1-ia, 2m+2; 2ix)$$

$$= \left(1 - ix + \frac{i^2x^2}{2!} + \cdots\right)\left(1 + \frac{m+1-ia}{2m+2}\frac{2ix}{1!} + \cdots\right)$$

$$= 1 + \frac{a}{m+1}x + \cdots$$

we have

$$c_0 = 1, \qquad c_1 = \frac{a}{m+1}$$

A sufficient number of terms was taken in order to insure an accuracy of about 8 significant figures in the computed values.

It is shown in the "supplementary note" at the end of this paper that the truncating error of the expansion (40) is essentially of the order of magnitude of the first term neglected.

The first derivatives were computed and checked by means of the formulae (35) and (37).

Method of Checking

The computed values of $H(m, a, x)$ were first tested by means of the recursion formula (39). An additional check is furnished by the agreement of the derivatives computed from (35) and (37).

The final manuscript values were subjected to a test based on the recursion formula (38) relating the functions and their corresponding first derivatives.

For the special case $H(m, 0, x)$ of the function $H(m, a, x)$, a powerful additional check was furnished by the agreement of the computed values with those obtained from (21) in terms of the "Spherical Bessel Functions" computed by this project.

Supplementary Note

From (40) it follows that

$$|R| = \left| H(m, a, x) - \sum_0^n \xi_k \right| = \left| \sum_{k=n+1}^{\infty} \xi_k \right| \lessgtr \sum_{k=n+1}^{\infty} |\xi_k| \tag{42}$$

In order to obtain an upper bound for $|R|$ we start with the recursion formula

$$(2m + 2 + k)c_{k+1} - 2ac_k + kc_{k-1} = 0$$

whence

$$U_{k+1}c_{k+1} = \frac{2ax}{(k+1)(2m+2+k)} c_k U_k - \frac{x^2}{(k+1)(2m+2+k)} c_{k-1} U_{k-1}$$

where

$$U_k = \frac{x^k}{k!}$$

With the definition of the ξ's from (40), the last equation yields

$$g_{k+1} \lessgtr \frac{2ax}{(n+1)(2m+2+n)} g_k + \frac{x^2}{(n+1)(2m+2+n)} g_{k-1}, \qquad \begin{matrix} k \lessgtr n \\ a \lessgtr 0 \end{matrix} \tag{43}$$

(where for the sake of brevity g_k was written for $|\xi_k|$)

Since the coefficients of (43) are independent of k, it may be verified that $\psi(x) = \sum_{n+1}^{\infty} g_k$ satisfies the inequality

$$\left\{1 - \frac{2ax}{(n+1)(2m+2+n)} - \frac{x^2}{(n+1)(2m+2+n)}\right\}\psi(x) \lesseqgtr g_{n+1} + g_{n+2} - \frac{2axg_{n+1}}{(n+1)(2m+2+n)}$$

If the expression in braces is positive, the last inequality yields

$$\psi(x) = \sum_{n+1}^{\infty} g_k \lesseqgtr g_{n+1}\left\{\frac{1 - \dfrac{2ax}{(n+1)(2m+2+n)} + \dfrac{g_{n+2}}{g_{n+1}}}{1 - \dfrac{2ax + x^2}{(n+1)(2m+2+n)}}\right\} = g_{n+1}E(\text{say})$$

Whence it follows that the remainder satisfies the inequality

$$|R| \lesseqgtr \psi(x) \lesseqgtr g_{n+1}E$$

where E is essentially of the order of unity for all cases of practical importance.

TABLE OF THE CONFLUENT HYPERGEOMETRIC FUNCTION $H(m, a, x)$ AND ITS FIRST DERIVATIVE $\frac{d}{dx} H(m, a, x) = H'(m, a, x)$

$$H(m, a, x) = e^{-ix} F(m + 1 - ia, 2m + 2; 2ix)$$

$$H(m, a, 0) = 1, \quad \left\{\frac{d}{dx} H(m, a, x)\right\}_{x=0} = \frac{a}{m + 1}$$

$a = 0$

x	$m = 0$		$m = 1$	
	$H(m, a, x)$	$H'(m, a, x)$	$H(m, a, x)$	$H'(m, a, x)$
0	1.000000 (0)	0.	1.000000 (0)	0.
1	8.414710 (−1)	−3.011687 (−1)	9.035060 (−1)	−1.861052 (−1)
2	4.546487 (−1)	−4.353978 (−1)	6.530967 (−1)	−2.976719 (−1)
3	4.704000 (−2)	−3.456775 (−1)	3.456775 (−1)	−2.986375 (−1)
4	−1.892006 (−1)	−1.161107 (−1)	8.708306 (−2)	−2.072128 (−1)
5	−1.917849 (−1)	9.508941 (−2)	−5.705364 (−2)	−8.083873 (−2)
6	−4.656925 (−2)	1.677899 (−1)	−8.389496 (−2)	1.866286 (−2)
7	9.385523 (−2)	9.429243 (−2)	−4.041104 (−2)	5.754269 (−2)
8	1.236698 (−1)	−3.364623 (−2)	1.261734 (−2)	4.164467 (−2)
9	4.579094 (−2)	−1.063246 (−1)	3.544153 (−2)	3.449806 (−3)
10	−5.440211 (−2)	−7.846694 (−2)	2.354008 (−2)	−2.338266 (−2)

$a = 1$

x	$m = 0$		$m = 1$	
	$H(m, a, x)$	$H'(m, a, x)$	$H(m, a, x)$	$H'(m, a, x)$
0	1.000000 (0)	1.000000 (0)	1.000000 (0)	5.000000 (−1)
1	2.098514 (0)	1.117925 (0)	1.470885 (0)	4.120052 (−1)
2	3.051864 (0)	6.948126 (−1)	1.767789 (0)	1.583244 (−1)
3	3.332803 (0)	−1.827238 (−1)	1.757763 (0)	−1.827238 (−1)
4	2.668086 (0)	−1.111412 (0)	1.417312 (0)	−4.792313 (−1)
5	1.263460 (0)	−1.587548 (0)	8.553024 (−1)	−6.104080 (−1)
6	−2.569936 (−1)	−1.329123 (0)	2.680323 (−1)	−5.305453 (−1)
7	−1.194160 (0)	−4.818188 (−1)	−1.526446 (−1)	−2.937192 (−1)
8	−1.191300 (0)	4.520452 (−1)	−3.081273 (−1)	−2.306262 (−2)
9	−4.451861 (−1)	9.319898 (−1)	−2.295293 (−1)	1.576437 (−1)
10	4.404628 (−1)	7.317549 (−1)	−4.369381 (−2)	1.889408 (−1)

$a = 2$

x	$m = 0$				$m = 1$			
	$H(m, a, x)$		$H'(m, a, x)$		$H(m, a, x)$		$H'(m, a, x)$	
0	1.000000	(0)	2.000000	(0)	1.000000	(0)	1.000000	(0)
1	4.365329	(0)	4.895832	(0)	2.300896	(0)	1.591508	(0)
2	1.090999	(1)	8.111260	(0)	4.112614	(0)	1.970831	(0)
3	2.007137	(1)	9.772662	(0)	6.074014	(0)	1.849324	(0)
4	2.927516	(1)	7.938480	(0)	7.591775	(0)	1.078986	(0)
5	3.454231	(1)	1.967653	(0)	8.054037	(0)	−2.151078	(−1)
6	3.235245	(1)	−6.504799	(0)	7.120970	(0)	−1.626200	(0)
7	2.189960	(1)	−1.386620	(1)	4.942748	(0)	−2.618275	(0)
8	6.295483	(0)	−1.627542	(1)	2.164979	(0)	−2.781019	(0)
9	−8.380400	(0)	−1.199878	(1)	−3.174681	(−1)	−2.052708	(0)
10	−1.603657	(1)	−2.830389	(0)	−1.754565	(0)	−7.754713	(−1)

$a = 0$

x	$m = 2$				$m = 3$			
	$H(m, a, x)$		$H'(m, a, x)$		$H(m, a, x)$		$H'(m, a, x)$	
0	1.000000	(0)	0.		1.000000	(0)	0.	
1	9.305258	(−1)	−1.350987	(−1)	9.456910	(−1)	−1.061567	(−1)
2	7.441798	(−1)	−2.277079	(−1)	7.969775	(−1)	−1.847920	(−1)
3	4.977292	(−1)	−2.534194	(−1)	5.913120	(−1)	−2.183600	(−1)
4	2.590160	(−1)	−2.149161	(−1)	3.761032	(−1)	−2.049027	(−1)
5	8.083873	(−2)	−1.378924	(−1)	1.930493	(−1)	−1.570948	(−1)
6	−1.555238	(−2)	−5.695215	(−2)	6.644418	(−2)	−9.566265	(−2)
7	−4.110192	(−2)	4.934837	(−4)	−4.934837	(−4)	−4.060844	(−2)
8	−2.602792	(−2)	2.415328	(−2)	−2.113412	(−2)	−4.282070	(−3)
9	−1.916559	(−3)	2.075449	(−2)	−1.614238	(−2)	1.106453	(−2)
10	1.169133	(−2)	5.924377	(−3)	−4.147064	(−3)	1.108687	(−2)

$a = 1$

x	$m = 2$		$m = 3$	
	$H(m, a, x)$	$H'(m, a, x)$	$H(m, a, x)$	$H'(m, a, x)$
0	1.000000 (0)	3.333333 (−1)	1.000000 (0)	2.500000 (−1)
1	1.293748 (0)	2.388079 (−1)	1.212381 (0)	1.654407 (−1)
2	1.451140 (0)	6.605204 (−2)	1.315633 (0)	3.572983 (−2)
3	1.415474 (0)	−1.372545 (−1)	1.279066 (0)	−1.080709 (−1)
4	1.187887 (0)	−3.071627 (−1)	1.107422 (0)	−2.283266 (−1)
5	8.304474 (−1)	−3.903686 (−1)	8.406524 (−1)	−2.945045 (−1)
6	4.430410 (−1)	−3.673610 (−1)	5.407934 (−1)	−2.943090 (−1)
7	1.242268 (−1)	−2.598787 (−1)	2.711589 (−1)	−2.373183 (−1)
8	−6.550052 (−2)	−1.188915 (−1)	7.643316 (−2)	−1.496697 (−1)
9	−1.210704 (−1)	2.802397 (−4)	−2.844593 (−2)	−6.255928 (−2)
10	−8.431508 (−2)	6.246817 (−2)	−5.706112 (−2)	−5.740307 (−5)

$a = 2$

x	$m = 2$		$m = 3$	
	$H(m, a, x)$	$H'(m, a, x)$	$H(m, a, x)$	$H'(m, a, x)$
0	1.000000 (0)	6.666667 (−1)	1.000000 (0)	5.000000 (−1)
1	1.773470 (0)	8.636616 (−1)	1.544234 (0)	5.751574 (−1)
2	2.677230 (0)	9.112321 (−1)	2.116770 (0)	5.504280 (−1)
3	3.520575 (0)	7.351564 (−1)	2.603829 (0)	4.031902 (−1)
4	4.070493 (0)	3.311095 (−1)	2.886554 (0)	1.475240 (−1)
5	4.134572 (0)	−2.151078 (−1)	2.880059 (0)	−1.637206 (−1)
6	3.644654 (0)	−7.477240 (−1)	2.566437 (0)	−4.530380 (−1)
7	2.700365 (0)	−1.098663 (0)	2.006936 (0)	−6.445272 (−1)
8	1.545624 (0)	−1.158528 (0)	1.325995 (0)	−6.918208 (−1)
9	4.820110 (−1)	−9.261661 (−1)	6.717344 (−1)	−5.953856 (−1)
10	−2.447734 (−1)	−5.101224 (−1)	1.681325 (−1)	−4.011225 (−1)

$a = 3$

x	$m = 0$				$m = 1$			
	$H(m, a, x)$		$H'(m, a, x)$		$H(m, a, x)$		$H'(m, a, x)$	
0	1.000000	(0)	3.000000	(0)	1.000000	(0)	1.500000	(0)
1	8.206035	(0)	1.298295	(1)	3.490546	(0)	3.674831	(0)
2	3.073645	(1)	3.414092	(1)	8.710265	(0)	6.908486	(0)
3	8.031294	(1)	6.648330	(1)	1.744555	(1)	1.053073	(1)
4	1.647780	(2)	1.017250	(2)	2.944567	(1)	1.316221	(1)
5	2.790362	(2)	1.226569	(2)	4.286711	(1)	1.310016	(1)
6	3.984571	(2)	1.089644	(2)	5.432035	(1)	9.107342	(0)
7	4.812390	(2)	4.880064	(1)	5.978213	(1)	1.277983	(0)
8	4.830630	(2)	−4.949888	(1)	5.620079	(1)	−8.529060	(0)
9	3.800404	(2)	−1.539065	(2)	4.313425	(1)	−1.710072	(1)
10	1.884041	(2)	−2.191462	(2)	2.353076	(1)	−2.113027	(1)

$a = 4$

x	$m = 0$				$m = 1$			
	$H(m, a, x)$		$H'(m, a, x)$		$H(m, a, x)$		$H'(m, a, x)$	
0	1.000000	(0)	4.000000	(0)	1.000000	(0)	2.000000	(0)
1	1.442783	(1)	2.843304	(1)	5.166757	(0)	7.116201	(0)
2	7.555192	(1)	1.064673	(2)	1.727121	(1)	1.833621	(1)
3	2.610544	(2)	2.853816	(2)	4.463742	(1)	3.786733	(1)
4	6.934283	(2)	6.040035	(2)	9.572249	(1)	6.538942	(1)
5	1.513151	(3)	1.051712	(3)	1.765020	(2)	9.598112	(1)
6	2.807014	(3)	1.526236	(3)	2.853477	(2)	1.194425	(2)
7	4.507111	(3)	1.823304	(3)	4.085330	(2)	1.224016	(2)
8	6.309383	(3)	1.689931	(3)	5.194324	(2)	9.350194	(1)
9	7.680626	(3)	9.449413	(2)	5.838738	(2)	3.008896	(1)
10	8.003283	(3)	−3.774662	(2)	5.715988	(2)	−5.688990	(1)

$a = 5$

x	$m = 0$		$m = 1$	
	$H(m, a, x)$	$H'(m, a, x)$	$H(m, a, x)$	$H'(m, a, x)$
0	1.000000 (0)	5.000000 (0)	1.000000 (0)	2.500000 (0)
1	2.416651 (1)	5.588186 (1)	7.494310 (0)	1.254504 (1)
2	1.694249 (2)	2.823392 (2)	3.258377 (1)	4.234284 (1)
3	7.430556 (2)	9.786791 (2)	1.052538 (2)	1.115328 (2)
4	2.447365 (3)	2.641371 (3)	2.767919 (2)	2.439701 (2)
5	6.552292 (3)	5.878050 (3)	6.203864 (2)	4.572114 (2)
6	1.487433 (4)	1.110437 (4)	1.216678 (3)	7.454334 (2)
7	2.934574 (4)	1.804353 (4)	2.121184 (3)	1.061746 (3)
8	5.105725 (4)	2.522259 (4)	3.318226 (3)	1.311005 (3)
9	7.894355 (4)	2.983126 (4)	4.678032 (3)	1.365013 (3)
10	1.086735 (5)	2.832281 (4)	5.942821 (3)	1.105086 (3)

$a = 6$

x	$m = 0$		$m = 1$	
	$H(m, a, x)$	$H'(m, a, x)$	$H(m, a, x)$	$H'(m, a, x)$
0	1.000000 (0)	6.000000 (0)	1.000000 (0)	3.000000 (0)
1	3.899737 (1)	1.021948 (2)	1.068563 (1)	2.082145 (1)
2	3.551659 (2)	6.732969 (2)	5.909588 (1)	8.952971 (1)
3	1.923993 (3)	2.913109 (3)	2.332662 (2)	2.911297 (2)
4	7.646266 (3)	9.631432 (3)	7.347196 (2)	7.753426 (2)
5	2.439775 (4)	2.607359 (4)	1.951020 (3)	1.761917 (3)
6	6.560835 (4)	5.999658 (4)	4.508832 (3)	3.496771 (3)
7	1.530871 (5)	1.198934 (5)	9.250529 (3)	6.141056 (3)
8	3.157474 (5)	2.104939 (5)	1.706747 (4)	9.600146 (3)
9	5.825451 (5)	3.259091 (5)	2.855280 (4)	1.334726 (4)
10	9.683280 (5)	4.429290 (5)	4.351653 (4)	1.634424 (4)

$a = 3$

x	$m = 2$				$m = 3$			
	$H(m, a, x)$		$H'(m, a, x)$		$H(m, a, x)$		$H'(m, a, x)$	
0	1.000000	(0)	1.000000	(0)	1.000000	(0)	7.500000	(−1)
1	2.401519	(0)	1.842855	(0)	1.955324	(0)	1.168040	(0)
2	4.736085	(0)	2.831321	(0)	3.333337	(0)	1.576282	(0)
3	8.019280	(0)	3.681534	(0)	5.060703	(0)	1.842643	(0)
4	1.192550	(1)	4.011972	(0)	6.924333	(0)	1.827701	(0)
5	1.575400	(1)	3.482105	(0)	8.590327	(0)	1.438816	(0)
6	1.855723	(1)	1.966764	(0)	9.677769	(0)	6.815960	(−1)
7	1.942752	(1)	−3.165566	(−1)	9.872038	(0)	−3.165566	(−1)
8	1.785197	(1)	−2.809943	(0)	9.039588	(0)	−1.328752	(0)
9	1.398326	(1)	−4.779903	(0)	7.296787	(0)	−2.096194	(0)
10	8.680985	(0)	−5.596591	(0)	4.997152	(0)	−2.418468	(0)

$a = 4$

x	$m = 2$				$m = 3$			
	$H(m, a, x)$		$H'(m, a, x)$		$H(m, a, x)$		$H'(m, a, x)$	
0	1.000000	(0)	1.333333	(0)	1.000000	(0)	1.000000	(0)
1	3.217314	(0)	3.312589	(0)	2.462451	(0)	2.000772	(0)
2	8.103110	(0)	6.714039	(0)	5.153535	(0)	3.452130	(0)
3	1.713584	(1)	1.156430	(1)	9.478128	(0)	5.230483	(0)
4	3.151389	(1)	1.723297	(1)	1.561490	(1)	7.003368	(0)
5	5.140458	(1)	2.228827	(1)	2.331059	(1)	8.250796	(0)
6	7.520881	(1)	2.469811	(1)	3.174373	(1)	8.384294	(0)
7	9.923775	(1)	2.244964	(1)	3.955225	(1)	6.949167	(0)
8	1.181704	(2)	1.444806	(1)	4.508041	(1)	3.846488	(0)
9	1.264065	(2)	1.335459	(0)	4.681784	(1)	−5.214780	(−1)
10	1.200087	(2)	−1.422247	(1)	4.390700	(1)	−5.271449	(0)

$a = 5$

x	$m = 2$				$m = 3$			
	$H(m, a, x)$		$H'(m, a, x)$		$H(m, a, x)$		$H'(m, a, x)$	
0	1.000000	(0)	1.666667	(0)	1.000000	(0)	1.250000	(0)
1	4.269470	(0)	5.450527	(0)	3.085623	(0)	3.144227	(0)
2	1.348848	(1)	1.401704	(1)	7.841424	(0)	6.695649	(0)
3	3.485097	(1)	3.021062	(1)	1.721650	(1)	1.245294	(1)
4	7.724304	(1)	5.632846	(1)	3.354284	(1)	2.057061	(1)
5	1.508627	(2)	9.236694	(1)	5.894981	(1)	3.042838	(1)
6	2.639382	(2)	1.341046	(2)	9.443587	(1)	4.035956	(1)
7	4.178536	(2)	1.720306	(2)	1.388096	(2)	4.769453	(1)
8	6.021172	(2)	1.922748	(2)	1.879007	(2)	4.927161	(1)
9	7.915760	(2)	1.802022	(2)	2.345188	(2)	4.240215	(1)
10	9.484114	(2)	1.261761	(2)	2.695121	(2)	2.604277	(1)

$a = 6$

x	$m = 2$				$m = 3$			
	$H(m, a, x)$		$H'(m, a, x)$		$H(m, a, x)$		$H'(m, a, x)$	
0	1.000000	(0)	2.000000	(0)	1.000000	(0)	1.500000	(0)
1	5.617714	(0)	8.486422	(0)	3.848609	(0)	4.686520	(0)
2	2.193948	(1)	2.707255	(1)	1.176449	(1)	1.208348	(1)
3	6.811149	(1)	7.092337	(1)	3.047315	(1)	2.687650	(1)
4	1.786020	(2)	1.593409	(2)	6.925208	(1)	5.285821	(1)
5	4.091143	(2)	3.145627	(2)	1.410265	(2)	9.327007	(1)
6	8.358104	(2)	5.534200	(2)	2.609135	(2)	1.488860	(2)
7	1.543609	(3)	8.741146	(2)	4.426208	(2)	2.157469	(2)
8	2.600142	(3)	1.241656	(3)	6.927599	(2)	2.834395	(2)
9	4.017286	(3)	1.578984	(3)	1.004203	(3)	3.351044	(2)
10	5.710268	(3)	1.772329	(3)	1.350749	(3)	3.501655	(2)

$a = 7$

x	$m = 0$				$m = 1$			
	$H(m, a, x)$		$H'(m, a, x)$		$H(m, a, x)$		$H'(m, a, x)$	
0	1.000000	(0)	7.000000	(0)	1.000000	(0)	3.500000	(0)
1	6.107818	(1)	1.773340	(2)	1.501279	(1)	3.310661	(1)
2	7.065692	(2)	1.486921	(3)	1.037719	(2)	1.777926	(2)
3	4.635380	(3)	7.842529	(3)	4.921027	(2)	6.985590	(2)
4	2.180695	(4)	3.098262	(4)	1.824990	(3)	2.211537	(3)
5	8.131856	(4)	9.922126	(4)	5.640104	(3)	5.926346	(3)
6	2.536738	(5)	2.690223	(5)	1.506694	(4)	1.383482	(4)
7	6.840178	(5)	6.339816	(5)	3.560694	(4)	2.864179	(4)
8	1.628640	(6)	1.320096	(6)	7.560291	(4)	5.316874	(4)
9	3.473836	(6)	2.452847	(6)	1.457600	(5)	8.903844	(4)
10	6.703153	(6)	4.086288	(6)	2.570147	(5)	1.347386	(5)

$a = 8$

x	$m = 0$				$m = 1$			
	$H(m, a, x)$		$H'(m, a, x)$		$H(m, a, x)$		$H'(m, a, x)$	
0	1.000000	(0)	8.000000	(0)	1.000000	(0)	4.000000	(0)
1	9.333223	(1)	2.955086	(2)	2.082227	(1)	5.095169	(1)
2	1.347539	(3)	3.095570	(3)	1.773402	(2)	3.365766	(2)
3	1.054542	(4)	1.956294	(4)	9.969296	(2)	1.573054	(3)
4	5.789106	(4)	9.075477	(4)	4.296620	(3)	5.822872	(3)
5	2.487029	(5)	3.375204	(5)	1.525018	(4)	1.807019	(4)
6	8.867462	(5)	1.056372	(6)	4.644306	(4)	4.860707	(4)
7	2.719737	(6)	2.866676	(6)	1.245575	(5)	1.157598	(5)
8	7.346857	(6)	6.877525	(6)	2.994077	(5)	2.475321	(5)
9	1.776587	(7)	1.477751	(7)	6.530741	(5)	4.796726	(5)
10	3.890303	(7)	2.867723	(7)	1.304063	(6)	8.471852	(5)

$a = 9$

x	$m = 0$				$m = 1$			
	$H(m, a, x)$		$H'(m, a, x)$		$H(m, a, x)$		$H'(m, a, x)$	
0	1.000000	(0)	9.000000	(0)	1.000000	(0)	4.500000	(0)
1	1.396812	(2)	4.766868	(2)	2.855285	(1)	7.640959	(1)
2	2.481499	(3)	6.147163	(3)	2.960913	(2)	6.132893	(2)
3	2.288724	(4)	4.593059	(4)	1.951885	(3)	3.368390	(3)
4	1.449755	(5)	2.469698	(5)	9.675087	(3)	1.439951	(4)
5	7.096537	(5)	1.052437	(6)	3.903254	(4)	5.107990	(4)
6	2.859800	(6)	3.748332	(6)	1.340846	(5)	1.560966	(5)
7	9.861356	(6)	1.153009	(7)	4.035999	(5)	4.209250	(5)
8	2.984945	(7)	3.130567	(7)	1.085394	(6)	1.017980	(6)
9	8.073401	(7)	7.615732	(7)	2.644101	(6)	2.233064	(6)
10	1.976336	(8)	1.677417	(8)	5.893758	(6)	4.478126	(6)

$a = 10$

x	$m = 0$				$m = 1$			
	$H(m, a, x)$		$H'(m, a, x)$		$H(m, a, x)$		$H'(m, a, x)$	
0	1.000000	(0)	1.000000	(1)	1.000000	(0)	5.000000	(0)
1	2.053404	(2)	7.485701	(2)	3.875744	(1)	1.121745	(2)
2	4.435971	(3)	1.174042	(4)	4.844449	(2)	1.082840	(3)
3	4.774492	(4)	1.026100	(5)	3.711279	(3)	6.920846	(3)
4	3.457961	(5)	6.329645	(5)	2.097770	(4)	3.383684	(4)
5	1.911702	(6)	3.059010	(6)	9.539413	(4)	1.358435	(5)
6	8.630771	(6)	1.226618	(7)	3.665422	(5)	4.666919	(5)
7	3.315926	(7)	4.228545	(7)	1.227611	(6)	1.408876	(6)
8	1.114140	(8)	1.283241	(8)	3.660209	(6)	3.805595	(6)
9	3.336957	(8)	3.485329	(8)	9.862787	(6)	9.316446	(6)
10	9.033719	(8)	8.573494	(8)	2.428624	(7)	2.086323	(7)

$a = 7$

x	$m = 2$				$m = 3$			
	$H(m, a, x)$		$H'(m, a, x)$		$H(m, a, x)$		$H'(m, a, x)$	
0	1.000000	(0)	2.333333	(0)	1.000000	(0)	1.750000	(0)
1	7.335159	(0)	1.271511	(1)	4.779591	(0)	6.736599	(0)
2	3.498285	(1)	4.953265	(1)	1.743037	(1)	2.076285	(1)
3	1.287799	(2)	1.548082	(2)	5.274557	(1)	5.434047	(1)
4	3.939556	(2)	4.099489	(2)	1.382961	(2)	1.247132	(2)
5	1.042567	(3)	9.485507	(2)	3.224093	(2)	2.559328	(2)
6	2.446507	(3)	1.954252	(3)	6.796515	(2)	4.754784	(2)
7	5.174259	(3)	3.627723	(3)	1.310516	(3)	8.058720	(2)
8	9.973652	(3)	6.110501	(3)	2.330098	(3)	1.251216	(3)
9	1.765709	(4)	9.368474	(3)	3.841721	(3)	1.781271	(3)
10	2.886086	(4)	1.306390	(4)	5.895726	(3)	2.318902	(3)

$a = 8$

x	$m = 2$				$m = 3$			
	$H(m, a, x)$		$H'(m, a, x)$		$H(m, a, x)$		$H'(m, a, x)$	
0	1.000000	(0)	2.666667	(0)	1.000000	(0)	2.000000	(0)
1	9.510999	(0)	1.851237	(1)	5.911903	(0)	9.428603	(0)
2	5.482124	(1)	8.701257	(1)	2.553545	(1)	3.440572	(1)
3	2.367318	(2)	3.200691	(2)	8.952779	(1)	1.047352	(2)
4	8.355593	(2)	9.840883	(2)	2.684123	(2)	2.767410	(2)
5	2.525326	(3)	2.623554	(3)	7.095092	(2)	6.501187	(2)
6	6.723783	(3)	6.204267	(3)	1.686591	(3)	1.379149	(3)
7	1.607018	(4)	1.321025	(4)	3.654685	(3)	2.669665	(3)
8	3.493009	(4)	2.557810	(4)	7.289102	(3)	4.748263	(3)
9	6.969359	(4)	4.532593	(4)	1.347487	(4)	7.792684	(3)
10	1.285020	(5)	7.377265	(4)	2.320638	(4)	1.182325	(4)

$a = 9$

x	$m = 2$				$m = 3$			
	$H(m, a, x)$		$H'(m, a, x)$		$H(m, a, x)$		$H'(m, a, x)$	
0	1.000000	(0)	3.000000	(0)	1.000000	(0)	2.250000	(0)
1	1.225370	(1)	2.635409	(1)	7.284900	(0)	1.292689	(1)
2	8.460256	(1)	1.480104	(2)	3.702904	(1)	5.542022	(1)
3	4.247131	(2)	6.340774	(2)	1.493478	(2)	1.944758	(2)
4	1.714022	(3)	2.238230	(3)	5.081716	(2)	5.857242	(2)
5	5.861954	(3)	6.791792	(3)	1.511170	(3)	1.557588	(3)
6	1.754055	(4)	1.818756	(4)	4.017310	(3)	3.725181	(3)
7	4.689999	(4)	4.373573	(4)	9.696423	(3)	8.114293	(3)
8	1.137144	(5)	9.558442	(4)	2.148641	(4)	1.624032	(4)
9	2.526899	(5)	1.914569	(5)	4.406988	(4)	3.005036	(4)
10	5.186773	(5)	3.534925	(5)	8.417776	(4)	5.161641	(4)

$a = 10$

x	$m = 2$				$m = 3$			
	$H(m, a, x)$		$H'(m, a, x)$		$H(m, a, x)$		$H'(m, a, x)$	
0	1.000000	(0)	3.333333	(0)	1.000000	(0)	2.500000	(0)
1	1.569475	(1)	3.683965	(1)	8.944956	(0)	1.743207	(1)
2	1.287870	(2)	2.452098	(2)	5.319748	(1)	8.723839	(1)
3	7.458687	(2)	1.213008	(3)	2.452995	(2)	3.503298	(2)
4	3.415945	(3)	4.872467	(3)	9.412455	(2)	1.193239	(3)
5	1.312027	(4)	1.667248	(4)	3.128240	(3)	3.561379	(3)
6	4.378267	(4)	5.005295	(4)	9.237041	(3)	9.513099	(3)
7	1.299226	(5)	1.344506	(5)	2.465708	(4)	2.307523	(4)
8	3.484483	(5)	3.276089	(5)	6.024631	(4)	5.135570	(4)
9	8.546472	(5)	7.312862	(5)	1.359887	(5)	1.056611	(5)
10	1.934000	(6)	1.506124	(6)	2.855543	(5)	2.020642	(5)

The numbers in parentheses indicate the power of 10 by which tabulated values are to be multiplied, e.g. $H(0, 0, 1) = 8.414710 \times 10^{-1} = .8414710$.

The tabulated values of $H'(m, a, x)$ may have a maximum error of 2 units in the last significant figure.

The tabulated values of $H(m, a, x)$ may have a maximum error of one unit in the last significant figure.

ON THE STABILITY OF PRINCIPAL PERIODIC ORBITS IN THE THEORY OF PRIMARY COSMIC RAYS[1]

By Jaime Lifshitz[2]

1. **Introduction.** The motion of a charged particle in the field of a magnetic dipole, as a function of an independent variable σ and of the dependent variables x, λ and φ, is defined by the equations[3]:

$$\frac{d^2x}{d\sigma^2} = X(x, \lambda) = \frac{1}{2}\frac{\partial p}{\partial x}$$
$$\frac{d^2\lambda}{d\sigma^2} = \Lambda(x, \lambda) = \frac{1}{2}\frac{\partial p}{\partial \lambda} \tag{1}$$

$$P(x, \lambda) = \left(\frac{dx}{d\sigma}\right)^2 + \left(\frac{d\lambda}{d\sigma}\right)^2 = e^{2x}/16\gamma_1^4 + 2e^{-x} - e^{-2x}\cos^2\lambda - \sec^2\lambda \tag{2}$$

$$\frac{d\varphi}{d\sigma} = \sec^2\lambda - e^{-x} \tag{3}$$

where λ and φ are the magnetic latitude and longitude of the particle, respectively, σ is a monotonic function of the time and x is defined by the relation $e^x = 2\gamma_1 r$. Here r is the distance from the dipole to the particle, measured in units of length which eliminate all physical factors from the equations of motion. This unit of length is due to Störmer and is called after him. Further, γ_1 is a constant of integration which appears when the equation governing the longitude of the particle is reduced to the first order, as in (3).[4] Equations (1) are independent of φ and govern the motion of the particle in its own meridian plane.

In the theory of primary cosmic rays the knowledge of periodic and symmetric solutions of equations (1) and (2) plays a very important role; a family of such solutions, the so-called principal periodic orbits,[5] enjoys a predominant position in the theory of primary cosmic rays developed by Lemaitre and Vallarta. Using the notation adopted by Baños,[6] this type of solution is of the form:

$$x = \xi(\sigma) = z_0 + z_2\cos 2\omega\sigma + \cdots + z_{2n}\cos 2n\omega\sigma + \cdots$$
$$\lambda = \eta(\sigma) = \mu_1\sin\omega\sigma + \mu_3\sin 3\omega\sigma + \cdots + \mu_{2n+1}\sin(2n+1)\omega\sigma + \cdots \tag{4}$$

[1] Paper presented at the cosmic ray meeting held in conjunction with the Inter-American Astrophysical Congress in Tonanzintla, Pue. and Mexico, D. F., February 17–25, 1942.

[2] Research Worker at the Institute of Physics, National University of Mexico, Mexico, D.F.

[3] Alfredo Baños, Jr., Jour. of Math. and Phys., 18, 212, 1939.

[4] A complete analysis of the equations of motion in the theory of primary cosmic rays is given by Alfredo Baños, Jr., Jour. of the Franklin Institute, 227, 623, 1939.

[5] M. S. Vallarta, "An Outline of the Theory of the Allowed Cone of Cosmic Radiation," Univ. of Toronto Studies, Appl. Math. Series, No. 3, 1938, p. 21. This valuable paper contains, besides a complete outline of the theory of the allowed cone, an excellent presentation of the definitions and indispensable fundamental principles.

[6] Alfredo Baños, Jr., "Calculation of a Family of Asymptotic Orbits in the Field of a Magnetic Dipole," Ph.D. thesis, M.I.T., 1938, p. 6.

For each value of Störmer's parameter $\gamma_1 > \gamma_1^*$ there are two possible solutions of this type. Because of their position relative to the dipole they are known as the internal and external orbits, respectively. For the value of $\gamma_1 = \gamma_1^* = 0.78855$ these two orbits coincide. As the computation of these orbits proceeded,

TABLE I

No.	γ_1	ω	Ω_1	Ω_2	$\left(\frac{\Omega_1}{\omega}\right)^2$	$\left(\frac{\Omega_2}{\omega}\right)^2$
1E	1.000,0000	86603	70711		66667	
2E	964,9530	87964	70217		63719	
3E	929,8980	89278	69192		60065	
4E	895,0500	90473	67329		55582	
5E	860,8460	91418	64048		49085	
6E	828,2450	91842	58047		39946	
7E	800,0000	91052	45147		24586	
8E	790,0000	89420	28021		9820	
9E	788,6828	88558	15581		3096	
10L	788,5526	88151	0	88151i	0	−1.00000
9I	788,6828	87722	15795i	71927i	−3242	−67231
8I	790,0000	86637	29351i	57286i	−11477	−43721
7I	800,0000	83338	52462i	30876i	−39628	−13733
6I	828,2450	78044		21300		7449
5I	860,8460	73617		29880		16474
4I	895,0500	69832		31774		20704
3I	929,8980	66546		31295		22116
2I	964,9530	63661		29629		21662
1I	1.000,0000	61104		27181		19787
11I	1.040,0000	58506		23561		16217
12I	1.100,0000	55121		15799		8215
13I	1.160,0000	52221	46179i	6042i	−78198	−1339
14I	1.250,0000	48566	21662i	26904i	−19895	−30687
15I	1.282,0500	47426	14821i	32605i	−9766	−47264
16I	1.303,2800	46712	9076i	37636i	−3775	−64917
17I	1.312,0700	46425	5167i	41258i	−1239	−78978
18I	1.313,3400	46387	4060i	42327i	−765	−83274
19L	1.313,5887	46376	0	46376i	0	−1.00000

Note: Numerical values are given in units of the fifth decimal, except in the case of γ_1 where they are in units of the seventh.

they were labelled 1, 2, 3, $\cdots$ etc. External orbits were assigned the prefix E and internal ones I. The orbit corresponding to γ_1^* has been called the limiting periodic orbit and is here denoted by L-10. The other limiting orbit separating internal from equatorial orbits is denoted by L-19. The correspondence between identification numbers and the values of γ_1 appears in Table I. The family of principal periodic orbits in the meridian plane is shown in Fig. 1.

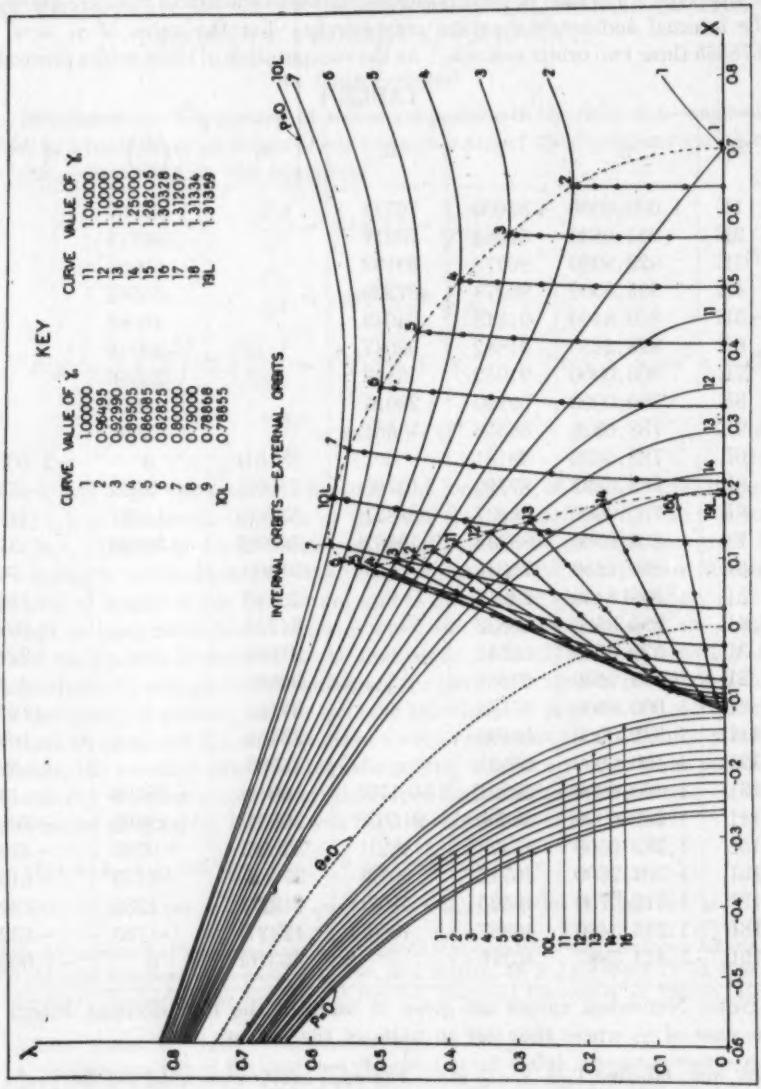

FIG. 1. The Family of Principal Periodic Orbits

As shown by Lemaitre and Vallarta, trajectories asymptotic to unstable principal periodic orbits separate orbits coming from infinity from those coming from some point on the earth. Tangents to asymptotic trajectories at a given point are the generators of the so-called allowed cone.[7] The determination of the characteristic exponents of principal periodic orbits is therefore a matter of very considerable interest, because these exponents are a measure of their stability or instability.[8] The characteristic exponent is found by studying the motion of the particle in the vicinity of a periodic orbit. More precisely, let

$$x = \xi + \delta x,$$

$$\lambda = \eta + \delta\lambda,$$

where ξ and η are the functions (4) mentioned above, and δx, $\delta\lambda$ are first order perturbations of the periodic motion. By virtue of (1) and (2) these perturbations must satisfy the so-called variational equations:

$$\begin{aligned} \frac{d^2(\delta x)}{d\sigma^2} &= (X_x)\delta x + (X_\lambda)\delta\lambda \\ \frac{d^2(\delta\lambda)}{d\sigma^2} &= (\Lambda_x)\delta x + (\Lambda_\lambda)\delta\lambda \end{aligned} \tag{5}$$

$$\frac{d\xi}{d\sigma}\frac{d(\delta x)}{d\sigma} + \frac{d\eta}{d\sigma}\frac{d(\delta\lambda)}{d\sigma} = (X)\delta x + (\Lambda)\delta\lambda \tag{6}$$

Parentheses mean that x and λ in each function must be substituted by the functions $\xi(\sigma)$ and $\eta(\sigma)$. The general solution of this system of equations is

$$\begin{aligned} \delta x &= a_1 e^{\Omega\sigma} F_1(\sigma) + a_2 e^{-\Omega\sigma} F_3(\sigma) + a_3 \frac{d\xi}{d\sigma} \\ \delta\lambda &= a_1 e^{\Omega\sigma} F_2(\sigma) + a_2 e^{-\Omega\sigma} F_4(\sigma) + a_3 \frac{d\eta}{d\sigma} \end{aligned} \tag{7}$$

where Ω is the characteristic exponent, $F_k (k = 1, 2, 3, 4)$ are periodic functions of σ with period $2\pi/\omega$ and a_i $(i = 1, 2, 3)$ are arbitrary integration constants. If Ω is real the orbit is unstable and the asymptotic trajectory going away from the orbit is defined by the conditions $a_2 = a_3 = 0$ (with $\Omega > 0$). For $\sigma = -\infty$ the asymptotic trajectory coincides with the periodic orbit.

2. Godart's method for the calculation of the characteristic exponent of external periodic orbits. External periodic orbits are unstable. Their characteristic exponents were calculated by Godart.[9] For this purpose he introduced normal and tangential displacements and reduced the variational equations

[7] G. Lemaître and M. S. Vallarta, Phys. Rev., 50, 493, 1936; also Vallarta, loc. cit., ref. 3.

[8] H. Poincaré, "Les méthodes nouvelles de la mécanique céleste," Paris, Gauthier-Villars, 1892, Vol. I, Chap. IV; G. Lemaître and M. S. Vallarta, Ann. de la Soc. Sci. de Bruxelles, 56, 115, 1936; G. Lemaître and M. S. Vallarta, Phys. Rev., 50, 497, 1936.

[9] O. Godart, Ann. de la Soc. Sci. de Bruxelles, 58, 27, 1938.

(5) and (6) to a single equation of the second order of Hill's type. Explicitly this equation may be written:

$$\frac{1}{\omega^2}\frac{d^2w}{d\sigma^2} + \left(\theta_0 + 2\sum_{k=1}^{\infty}\theta_{2k}\cos 2k\omega\sigma\right)w = 0.$$

As shown by Godart, it has the same characteristic exponent as the solutions of the general system. To find the characteristic exponent, the value of a determinant of infinite order must be calculated for a value of the unknown Ω (usually for $\Omega = 0$, as was done by Godart) and the problem is reduced to a simple relation which gives the value of Ω for which the determinant vanishes. The derivation of the value of Hill's determinant for $\Omega = 0$ was carried out by Godart by expanding the determinant in a series. This series converges rapidly for external periodic orbits which are far from the limiting periodic orbit separating internal from external orbits, but as the limiting orbit itself is approached, the second harmonic in Hill's coefficient θ_2 increases in absolute value and the series converges less rapidly. For internal orbits this disadvantage becomes more serious, and the method is practically unusable beginning with orbit *I*-5. (See Table I.)

3. **Extension of Godart's method to internal orbits.** Since the Fourier expansions of the periodic orbits used by Godart are correct only to four decimals,[10] his results were first verified and then corrected to seven decimals in our Institute of Physics, using his own method. For external periodic orbits, real values of the characteristic exponents were obtained, since they are unstable. The characteristic exponent of the limiting periodic orbit vanishes, and internal periodic orbits have imaginary exponents, showing that they are stable.[11]

Applying Godart's method we obtained for orbit *I*-6 (Table I) the result

$$\sin^2\frac{\pi}{2}\nu > 1$$

where $\nu = \Omega/i\omega$, i.e. that the characteristic exponent is a complex number. From this we drew at first sight the wrong inference that this value was erroneous and that Godart's series expansion was already divergent. When this question was studied more carefully it was noted that in this case $\nu = 1 \pm i\nu_1$, where ν_1 is a real number. For the characteristic exponent Ω the expression is obtained

$$\Omega = i\omega \mp \omega\nu_1 = i\omega \mp \Omega_1$$

Substituting this in the variational equations we have a solution of the form

$$\delta x = e^{\Omega\sigma}F_1(\sigma) = e^{i\omega\sigma+\Omega_1\sigma}F_1(\sigma) = e^{\Omega_1\sigma}\Phi_1(\sigma)$$

$$\delta\lambda = e^{\Omega\sigma}F_2(\sigma) = e^{i\omega\sigma+\Omega_1\sigma}F_2(\sigma) = e^{\Omega_1\sigma}\Phi_2(\sigma)$$

[10] Alfredo Baños, Jr., Ingeniería, 13, 133, 1939.

[11] Alfredo Baños, Jr., Héctor Uribe and Jaime Lifshitz, Rev. of Modern Physics, 11, 137, 1939.

where the functions Φ_1 and Φ_2 are periodic, as well as F_1 and F_2, since the factor $e^{i\omega\sigma}$ has the same period $2\pi/\omega$ as F_1 and F_2.

It is therefore apparent that the orbit is unstable, since Ω_1 is real and positive. The only difference in the structure of the solution introducing the factor $e^{i\omega\sigma}$ is that even harmonics become odd, and vice versa. For external orbits, the periodic factor of δx has only even harmonics, and the corresponding factor of $\delta\lambda$ has only odd harmonics. For internal stable orbits the same structure of the factor which does not depend on the characteristic exponent may be adopted, and the value of the latter tends to zero as the orbit in question approaches the limiting periodic orbit. In the case of stable orbits the solution may take another form which has the characteristic exponent $\Omega_1 = i\omega - \Omega$ and changes the parity of harmonics in δx and $\delta\lambda$. As the internal periodic orbit approaches the limiting orbit its characteristic exponent Ω_1 tends to $i\omega$. It was found that as the value of Störmer's parameter tends to a value in the vicinity of $\gamma_1 = 0.812$ the value of Ω_1 tends to zero. For values of $\gamma_1 > 0.812$ the characteristic exponents again become real and the corresponding internal periodic orbits are unstable.

4. **Calculation of the characteristic exponent by means of the secular determinant.** Godart's method, as already pointed out, is inadequate to calculate the characteristic exponent of internal orbits, so that the method of the secular determinant must be resorted to. Expressing $F_1(\sigma)$ and $F_2(\sigma)$ as a Fourier series, the system of solutions takes the form:

$$\delta x = e^{\Omega\sigma}\Sigma(y_k' \sin k\omega\sigma + z_k' \cos k\omega\sigma),$$

$$\delta\lambda = e^{\Omega\sigma}\Sigma(\mu_k' \sin k\omega\sigma + \nu_k' \cos k\omega\sigma).$$

For external orbits, k has only even values in δx, and only odd values in $\delta\lambda$. On substituting these expressions in the variational equations one obtains an infinite system of linear and homogeneous equations in the unknowns y', z', μ', ν'. By an extension of the theorem on the compatibility of systems of linear and homogeneous equations, the determinant of the system must vanish, which yields a condition equation for Ω. The infinite determinant obtained in this way is called the secular determinant. Bearing in mind the definition of characteristic exponent,[10] it is clear that if Ω satisfies the condition of being one such exponent, so does every value $\pm\Omega + ik\omega(k = 0, \pm 1, \pm 2, \cdots)$. For unstable orbits there is a real value of the characteristic exponent; all other values satisfying the same condition are complex numbers. For the other possible case, that of stable orbits, all values of the characteristic exponent are pure imaginaries, two of which have an absolute value less than ω and satisfy the condition that their sum is equal to $i\omega$. To find one or the other of these values one must assume either that δx has even harmonics and $\delta\lambda$ odd ones, or conversely, thus obtaining two different secular determinants whose solutions are complementary. Baños and Godart made the first assumption. For unstable orbits only one of these secular determinants yields a real value of Ω; the other gives its comple-

ment to $i\omega$ which is a complex number. This fact is quite clear for orbit I-6 where, as already pointed out in Sect. 3, the calculation was made starting from the inadequate assumption that δx has even harmonics and $\delta\lambda$ odd ones. We obtained, as a result, a complex characteristic exponent.

For purposes of numerical calculation the secular determinant is brought down to a finite order by neglecting rows and columns above a certain index. The expression thus obtained is an algebraic equation in Ω^2. One assumes a value Ω_1 and substitutes it in all the elements of the determinant except in that of the first row and column. The condition that the determinant vanish is then a linear algebraic equation in Ω^2, and yields a value Ω_2. Repeating the same operation on Ω_2 yields a new value Ω_3. The sequence $\Omega_1, \Omega_2, \cdots \Omega_n \cdots$ tends to the desired characteristic exponent. To improve the convergency, instead of using the Ω_{i+1} found according to the procedure already outlined by the method of successive approximations, one may compare the results for two values Ω_i, Ω_j and compute the discrepancies $\Delta_i = \Omega_{i+1} - \Omega_i$, $\Delta_j = \Omega_{j+1} - \Omega_j$ thus obtaining

$$\Omega'_{i+1} = \frac{\Omega_i \Delta_j - \Omega_j \Delta_i}{\Delta_j - \Delta_i}$$

which is the value now introduced in the determinant, instead of Ω_{i+1}, continuing the process as in the method of successive approximations.

For the calculation of Ω by the method of the secular determinant, instead of expressing the latter as the sum of several determinants without the desired unknown, one of which is multiplied by Ω^2, the elements of a row (or column) may be reduced to unity or zero by multiplying the columns (or rows) respectively by the reciprocal of the element. The numerical factors appearing in the course of this process are omitted, because the condition that the original determinant vanish demands that the derived one also vanish, since it is a factor of a vanishing product and none of the other factors vanishes. Subtracting rows, all the elements of a selected column may be made to vanish except one. The condition to be satisfied by the determinant demands that the minor belonging to this element also vanish, thus yielding a determinant of lower order. The process may be continued until a determinant of sufficiently low order (second or third) is obtained so that its direct computation is feasible. The method differs from the standard method for the computation of determinants of higher order insofar as the factors appearing when the elements of a column become unity are cancelled, which in turn simplifies the calculation and reduces rounding errors appearing in the products. In addition only one determinant is actually computed instead of several.

5. **Results of the calculation of the characteristic exponent for the complete family of principal external and internal periodic orbits.** The results obtained by applying the methods outlined above are contained in Table I. Here Ω_1 is the characteristic exponent obtained when δx contains even harmonics and $\delta\lambda$ odd ones, while Ω_2 is the characteristic exponent for the converse case. Ob-

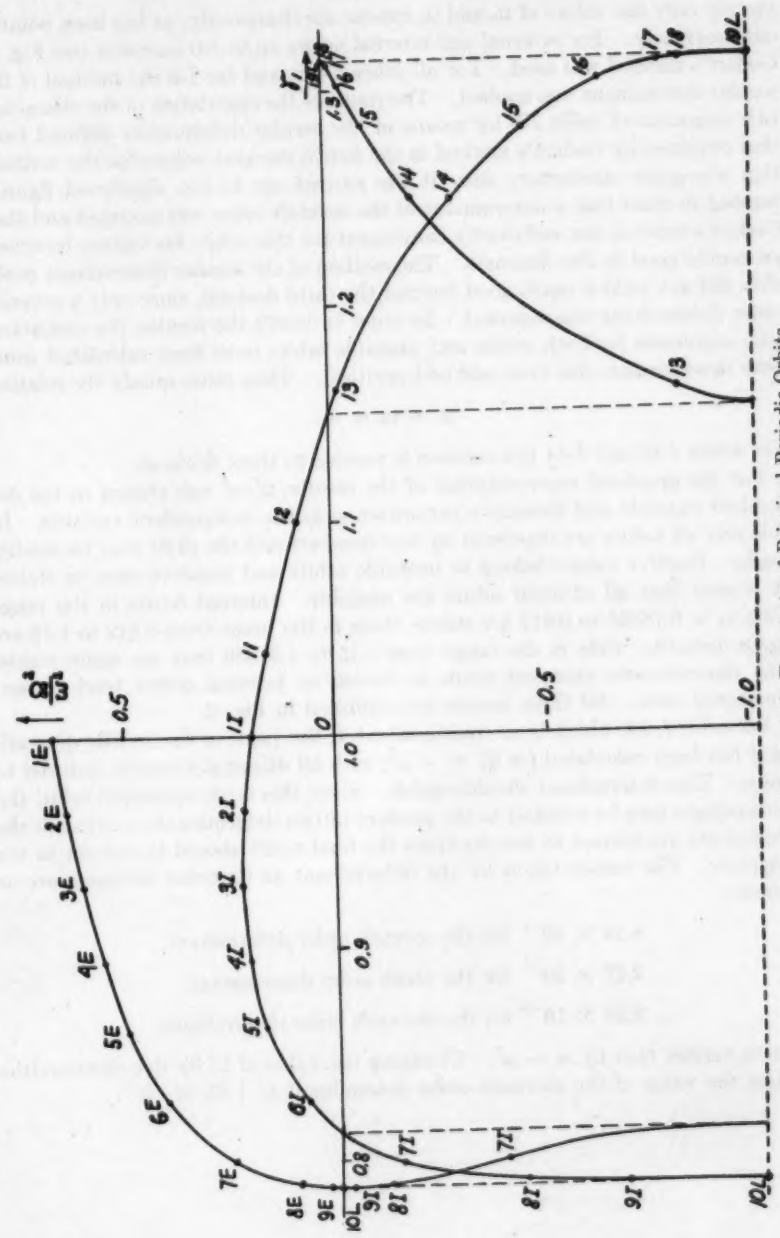

FIG. 2. Characteristic Exponents of Principal Periodic Orbits

viously only the values of Ω_1 and Ω_2 appear simultaneously, as has been pointed out previously. For external and internal orbits up to I-6 inclusive (see Fig. 1) Godart's method was used. For all other orbits and for I-6 the method of the secular determinant was applied. The result of the calculation of the characteristic exponent of orbit I-6 by means of the secular determinant differed from that obtained by Godart's method in the fourth decimal, otherwise the verification was quite satisfactory since it was carried out to five significant figures, bearing in mind that a determinant of the seventh order was assumed and that Godart's series is not sufficiently convergent for this orbit; his values, however, are surely good to five decimals. The method of the secular determinant probably did not yield a result good beyond the third decimal, since only a seventh order determinant was assumed. In order to verify the results, the characteristic exponents for both stable and uhstable orbits have been calculated using both determinants (for even and odd parities). They must satisfy the relation

$$\Omega_1 + \Omega_2 = i\omega.$$

For orbits I-13 and I-14 this relation is verified to three decimals.

For the graphical representation of the results, Ω^2/w^2 was chosen as the dependent variable and Störmer's parameter γ_1 as the independent variable. In this way all values are expressed by real numbers and the plots may be readily made. Positive values belong to unstable orbits and negative ones to stable. It is seen that all external orbits are unstable. Internal orbits in the range from $\gamma_1 = 0.78855$ to 0.812 are stable, those in the range from 0.812 to 1.15 are again unstable, while in the range from 1.15 to 1.31359 they are again stable. The characteristic exponent tends to vanish as internal orbits tend toward equatorial ones. All these results are exhibited in Fig. 2.

For orbit L-19, which is an equatorial orbit, the value of the secular determinant has been calculated for $\Omega_2^2 = -\omega^2$, with all diagonal elements reduced to unity. This determinant should vanish. Since this is an equatorial orbit, the determinant may be reduced to the product of two determinants, so that, if the coefficients are correct to five decimals the final result should be correct to ten decimals. The values taken by the determinant as its order increases are as follows:

8.43×10^{-5} for the seventh order determinant,

2.27×10^{-7} for the ninth order determinant,

3.23×10^{-10} for the eleventh order determinant,

which verifies that $\Omega_2^2 = -\omega^2$. Changing the value of Ω_2^2 by five thousandths raises the value of the eleventh order determinant to 1.33×10^{-7}.